Tools for the Rectilinear Steiner Tree Problem

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RUU-CS-90-8
March 1990

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ISSN:0924-3275

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April 4, 1990

Abstract

A Minimal Rectilinear Steiner Tree (RMST) for a set V of n points in the plane is a tree which connects the points of V using line segments parallel to the horizontal and vertical coordinate axes that has the shortest possible total length.

In this article, techniques are presented identifying connections that will be used in some RMST. These connections serve as a basis for an approximation algorithm. The identification methods are called contraction and clustering. Contractions identify line segments connected to the points on the enclosing rectangle of V and are totally independent of the further construction of the RMST. Clustering identifies connections that can also be internal to the enclosing rectangle, but which are subject to restrictions posed by the further construction of the RMST.

The approximation algorithm is a modified version of the Rectilinear Minimal Spanning Tree (RMSpT) algorithm, which uses dynamized Rectilinear Voronoi Diagrams defined on points and line segments. Even without contractions or clustering, it produces an approximation of shorter total length than the Minimal Rectilinear Spanning Tree, improving on a result by Hwang. Furthermore, the worst-case situation where the approximation based on MRSpTs could have a length of $\frac{3}{2}$ times the length of the RMST is identified.

The approximation yields minimal Steiner trees for any set V of $n \leqslant 5$ points, and trees with a length that is strictly less than $\frac{3}{2}$ times the length of the RMST for larger n. The complexity of the constructing algorithms is $O(n^2)$.

Acknowledgement

The authors wish to thank Dr. M. Overmars and especially Dr. M. Veldhorst for their continuous support. Dr. M. Veldhorst improved our proof of the Complex Contraction Theorem using induction.

1 Introduction

Let V be a finite set of n points in the plane. A Rectilinear Steiner Tree (RST) for V is a tree structure, composed solely of horizontal and vertical line segments, which interconnects all the points in V. A Minimal RST (RMST) for V is one in which the line segments used have the shortest possible total length. In contrast to the usual notion of a spanning tree, a RST is permitted to have three or more line segments meeting at a point that does not belong to V, called a Steiner point.

Applications of RST's can be found in wire layout for VLSI and printed circuit boards.

Garey and Johnson [GarJo77] proved the problem to be NP-complete. As a consequence, polynomial algorithms can only be expected to obtain results with a constant performance ratio. The Rectilinear Steiner Ratio (RSR) for a set V of points in the plane is defined as $\frac{|A|}{|T|}$, where $|A|$ denotes the length of an approximation for a RMST T for V. As was shown by Hanan [Hanan66], the problem is easily solved for $n \leqslant 4$. Hanan also mentioned the existence of an algorithm to implement his ideas, and conjectured that the case $n = 5$ is solvable without the use of exhaustive search. A carrier [Hanan66] is a line (horizontal or vertical) through one or more points of V. Hanan proved that for every set V, a RMST T exists whose line segments overlap some carrier, which reduces the number of RMSTs possible for V. Hwang [Hwang76] proved the RSR to be at most $\frac{3}{2}$ when $A = M$, a Rectilinear Minimal Spanning Tree (RMSpT) for V. In [Hwang79], it is shown that the Rectilinear Voronoi Diagram (RVD) for V, and subsequently M, can be computed in $O(nlogn)$ time. Kou et al. [KouMaBe81] obtained a RSR of $2(1 - \frac{1}{\lambda})$, in which λ denotes the number of leaves in a RMST, which is probably unknown. Their algorithm has been improved by Mehlhorn [Mehlhorn88]. For other heuristics, the authors refer to [Richards89] and [Bern84]. Polynomial-time special case algorithms have been designed by [AGH77] and [GeorPap87]. As for exhaustive search, the best algorithm known to the authors has a complexity of $O(n^{\sqrt{(n)}logn})$ [ThomDeSh87].

In this paper we will give a $O(n^2)$ approximation which yields RMST's for all sets V of $n \leqslant 5$ points and which yields trees with a RSR of less than $\frac{3}{2}$ for larger n. Furthermore techniques identifying connections that will be used or need (can) not be used in some RMST, will be presented. These connections serve as a basis for an approximation. The identification methods are called contraction and clustering. Contractions identify line segments connected to the points on $\Re(V)$, the smallest enclosing rectangle of V, and are totally independant of the further construction of the RMST. Reduce Carriers is a technique which brings a certain part of a RST A for V into a canonical form. The specified part of A will then consist of vertical (or analogously horizontal) line segments only (with possibly one exception). This canonical form will serve as a basis for the transformations applied in the correctness proofs of the various clusterings. Clustering certifies that the path from vertex p_1 to vertex p_2, denoted $p_1 \bullet\!\!\leadsto\!\!\bullet p_2$, can be embedded within $\Re(\{p_1, p_2\})$. The RICH(V) defines a Convex Hull-like area for V. Proof will be given that a RMST for V exists embedded in RICH(V).

The paper is based on the following techniques: a dynamized version of Rectilinear Generalized Voronoi Diagrams in combination with RICH and the mentioned contraction and clustering techniques.

In this paper, the following terminology will be used: E denotes the set of line segments used in a tree. $\#V$ denotes the number of elements in V. A vertex in a RST T for V is a node of T, which is an element of V. A Steiner point in a RST T for V is a node in T with degree $\geqslant 3$, which is not in V. A corner point in a RST T for V is a node of T with degree 2, which is not in V. A virtual point in a RST T for V is a Steiner point or a corner point. A leaf of a RST for V is a node with degree one. Necessarily a leaf in a RMST is a vertex. An edge is a connection between two points, that can be virtual, containing either one line segment or two perpendicular line segments, which share a corner point. A line segment between two points, for example p_1 and p_2, will be denoted by $p_1 \bullet\!\!-\!\!\bullet p_2$. A path between p_1 and p_2 will be denoted by $p_1 \bullet\!\!\leadsto\!\!\bullet p_2$. A L-shape is an edge consisting of two perpendicular line segments. A forest for a set V of vertices is a set of pairwise disjoint RST's for subsets of V. $\Re(F)$ is the smallest enclosing rectangle of a forest F, $W(F)$ is the width of $\Re(F)$ and $\Gamma(F)$ is the height of $\Re(F)$. Let R be the number of contractions performed on V.

2 Contractions

2.1 Introduction

In this section the concept of contractions is introduced. Proof will be given that all RMSTs for a set V of points satisfying certain conditions, consist of a RMST for a reduced set $V' \subseteq V$ and a unique set of line segments connecting the RMST for V' to the points in $V - V'$. The NP-completeness of the RMST problem implies that it is resistant to a recursive approach. Nevertheless, the RMST for V can be shown not to contain any line segment in p, so the problem is reduced to a RMST problem for a smaller set of points V', where $\#V' \leqslant \#V$, and its search space is reduced.

Definition 1 *A* contraction *is a transformation of the set of points V into a set V' by projecting a set $p \subseteq V$ of one or more points on an outermost carrier on the carrier q adjacent to and parallel with p.*

Contractions do not depend on the grid in which V is placed, but solely on V. So an algorithm can be produced that is proportional in time to $\#V$ instead of proportional to the grid size. These transformations (possibly) reduce the search space and the number of points in a provable save way.

An algorithm implementing contractions and routing the reduced set V' consists of two stages. The first one contracts the collection of points, using minimal line length. The purpose of contractions is, to present a reduced set with the least number of points possible to the second stage, which will then try to find a near-optimal solution. Ideally, the reduced set of points contains only one element, for this can be handled trivially.

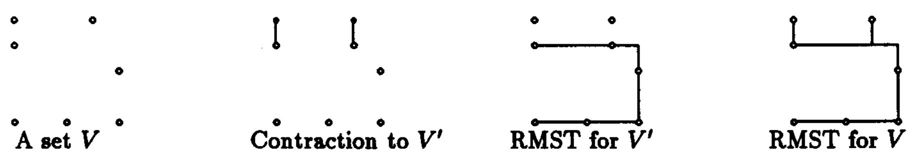
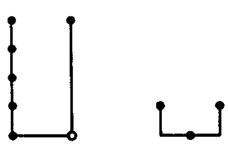


Figure 1: Contraction principle

To maintain minimality, the newly formed set V' of points must have a RMST T', which can be covered by some RMST T for V that also covers the paths over which the points of V were moved to their new locations entirely. We refer to figure 1 for a brief explanation. Here, a downward contraction is performed. Notice the leftmost point clashing with the one immediately below it, while the right one is moved to a grid point not in V.

Property 1 *When two points clash during a contraction, one of them can safely be deleted during future operations.*

Returning to figure 1 and comparing the RMSTs for the set V of points and its subset V', it is immediately clear that the performed contraction was a legal one; the line segments drawn with the contraction can be extended into a RMST for V.



The leftmost picture shows a situation that is definitely not optimal. It was constructed using downward contractions only. The open dot is not in V, but is used as intermediate point during some stage of the contraction. The rightmost picture shows the mistake made even clearer. A T–shape uses less line length than the proposed U–shape.

In the following, a simple rule will be presented that distinguishes legal contractions from illegal ones. Unfortunately, this rule restrains the possibility of contraction more than would be desirable. For example, the contraction performed in figure 1 is a legal one, but does not satisfy the conditions of the rule, and is therefore marked as illegal.

The contractions are classified according to their complexity: Simple contractions that affect exactly one point, Complex contractions that affect more than one point, and Global contractions, an extension to Complex contractions but affecting exactly two points.

2.2 Simple contractions

Definition 2 (Simple contraction) *Let V be a set of points, $\#V > 1$. Then point p of V is* simple contractable *if p is a unique extreme with respect to its x-, or y-coordinate. Let c be the nonempty carrier that would be extreme in the same sense as p if p would not exist. A simple contraction of a simple contractable point p is the projection of point p over one of p's carriers to c.*

Lemma 1 (Simple contraction) *Let point $t \in V$ be simple contractable to the carrier q. Then there is a RMST T for V that contains line segment t•–•u, where u is the orthogonal projection of t on q.*

Proof: This follows from the Complex Contraction Theorem in the next subsection. □

Corollary 1 *If t is a simple contractable point of a set V, then t is a leaf in all Rectilinear Minimal Steiner Trees T for V.*

Definition 3 *A set V is* Simple Contractable *if and only if use of the simple contraction theorem applies and its application produces a Simple Contractable set V' or a set V' where $\#V' = 1$.*

Let U(V) denote the shortest possible total length of a Rectilinear Steiner Tree for V. (This is an abstract notion and does not depict any particular tree.)

Lemma 2 (Simple Reduction) *Apply simple contractions as many times as possible on V, yielding a set V' of points and a set E of line segments. Then:*

1. $U(V) = U(V') + \sum_{l \in E} length(l)$.
2. $\#V' \leqslant \#V$.
3. *Either $\#V' = 1$ or each side of $\Re(V')$ contains at least 2 points of V'.*

Proof:

1. This follows from the Simple Contraction Lemma.
2. A point is either contracted onto another point, thus reducing the number of points by one, or it does not collide with another point, so the number of points remains the same. This reasoning holds for all contracted points.
3. If V is a simple contractable set, then $\#V' = 1$, otherwise ($\#V' > 3$) suppose that $\Re(V')$ has a side containing exactly one point. Then V' is contractable, contradicting the assumption that as many simple contractions as possible have been executed.

□

Corollary 2 *If V is a non-contractable set, $\#V \geqslant 1$, and s is a side of $\Re(V)$, then $\#V \geqslant 4$, $\#s \cap V \geqslant 2$ and V has at least two horizontal and two vertical nonidentical carriers.*

2.3 Complex Contractions

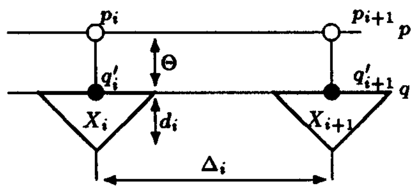
The conditions that a contraction R must satisfy to be **legal** will be defined. Informally, a legal contraction R of a set V of points is a contraction of V that does not affect the minimality of the RMST T for V if T is constructed by combining the tree T', which is a RMST for V', the set of points after the contraction, with the line segments between the original locations of the contracted points and their new locations.

Let p be an extreme carrier. In the following the direction of the contraction is denoted by one of the arrows $\downarrow, \leftarrow, \uparrow$ or $\rightarrow$. For example, if the contraction direction is downward, denoted by $\downarrow$, then p is the uppermost carrier. The black triangle ◀ is used to impose an ordering depending on the direction of the contraction. In the definitions and the theorems, it is assumed that p is the uppermost carrier, as depicted below. Let q denote the carrier parallel and adjacent to p.

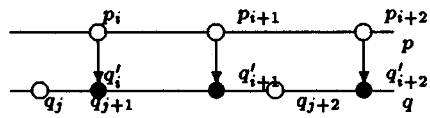
π (κ) denotes the number of vertices on p (q). For all $i \in \{1, \cdots, \pi\}$, q'_i denotes the projection of p_i on q. X_i denotes the triangle below or on q covering the area reachable using a path from p_i of length less than or equal to $min(\frac{\Delta_{i-1}}{2}, \frac{\Delta_i}{2})$.

Formally,

$$
\begin{array}{lcl}
V & := & \{v_1, v_2, \cdots, v_n\} \in \mathbf{R}^2; n \in \mathbf{N} \\
dir & \in & \{\downarrow, \leftarrow, \uparrow, \rightarrow\} \\
\blacktriangleleft & := & \left\{ \begin{array}{ll} dir = \downarrow & \Rightarrow (w \blacktriangleleft v \Leftrightarrow w_y \leqslant v_y) \\ dir = \leftarrow & \Rightarrow (w \blacktriangleleft v \Leftrightarrow w_x \leqslant v_x) \\ dir = \uparrow & \Rightarrow (w \blacktriangleleft v \Leftrightarrow w_y \geqslant v_y) \\ dir = \rightarrow & \Rightarrow (w \blacktriangleleft v \Leftrightarrow w_x \geqslant v_x) \end{array} \right. \\
p & := & \{v \in V \mid \forall_{w \in V} w \blacktriangleleft v\} \\
\pi & := & \#p \in \{1 \cdots n-1\} \\
p & := & \{p_1, p_2, \cdots, p_\pi\} \text{ (clockwise)} \\
q & := & \{v \in V - p \mid \forall_{w \in V-p} w \blacktriangleleft v\} \\
\kappa & := & \#q \in \{1 \cdots n-1\} \\
q & := & \{q_1, q_2, \cdots, q_\kappa\} \text{ (clockwise)} \\
v & := & (v_x, v_y) \\
w & := & (w_x, w_y) \\
\langle v, w \rangle & := & \mid v_x - w_x \mid + \mid v_y - w_y \mid \\
q & = & \varnothing \Rightarrow illegal(C, dir) \\
l_i & := & p_i \bullet\!\!-\!\!\bullet q'_i \\
L & := & \cup_{i=1}^{\pi} l_i
\end{array}
$$



$d_i = \frac{\Delta_i}{2} - \Theta$



Without loss of generality, assume $dir = \downarrow$.
For $i \in \{1 \cdots \pi\}$, define:

$$
\begin{array}{lcl}
q' & := & \{(p_{i_x}, q_{*_y}) \mid i \in \{1 \cdots \pi\}\} \\
\Delta_i & := & p_{i+1_x} - p_{i_x} \\
\Theta & := & p_{*_y} - q_{*_y} \\
x_i & \in & X_i \\
d_0 & := & \infty \\
d_\pi & := & \infty \\
d_i & := & \frac{\Delta_i}{2} - \Theta \text{ for } i \in \{1..\pi - 1\} \\
X_i & := & \{v \in V \mid v \blacktriangleleft q \wedge \langle v, q'_i \rangle \leqslant min(d_{i-1}, d_i)\}
\end{array}
$$

In the rest of this section, x_i will denote a point chosen arbitrarily from X_i. This choice remains invariant throughout the construction of the tree.

So, for a direction *dir*, call the outermost carrier p and the one alongside p q. The projection of p on q forms q', and the line segment $p_i \bullet\!\!-\!\!\bullet q'_i$ is called l_i. For all pairs (p_i, p_{i+1}) on p, the triangular areas X_i are defined depending on Δ_i, the distance between the pair of points on p, and Θ, the distance between the p- and q-carrier.

The total length of line on p used by a RST T is defined as $pl(T) := \mid p \cap T \mid$.

Definition 4 *A RST T for V is in canonical form with respect to the direction* dir, *denoted CF(T,dir), if p, the outermost carrier with respect to dir, does not contain any line segment or corner of an L-shape that can be shifted or reversed to q, the nearest carrier parallel to p.*

The canonical form of the RST T is denoted by $\mathsf{CF}(T, dir)$.
Because only shifts and reversions are used to create $\mathsf{CF}(T, dir)$, $\mid \mathsf{CF}(T, dir) \mid \leqslant \mid T \mid$ and $pl(\mathsf{CF}(T, dir)) \leqslant pl(T)$.

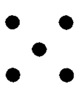
Corollary 3 *Let T' be a RST for the set V and p an extreme carrier, say upper, If* $\text{pl}(T') > 0$ *and there is a subscript i such that $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ and $p_{i+1} \bullet\!\!-\!\!\bullet p_{i+2}$ both have 2 downward connections, of which none overlap, then T' is not a RMST for V.*

Proof. Trivial, as $\mid \mathsf{CF}(T', dir) \mid < \mid T' \mid$. □

Corollary 4 *Let T' be a RST for the set V and p an extreme carrier, say upper, If $\mathrm{pl}(T') > 0$ and there is a subscript i such that $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ supports more than 2 downward connections, of which none overlap, then T' is not a RMST for V.*

Theorem 1 (Complex Contraction Theorem) *Let T' be a RST for the set V such that $CF(T', dir)$ and $\mathrm{pl}(T') > 0$ and $(\forall_{i \in \{1 \cdots \pi\}}\ X_i \neq \varnothing) \wedge q \neq \varnothing$. Then T' is not a RMST.*

First the above conditions will be shown a little too strong: there are sets V that do not satisfy the conditions of the Complex Contraction Theorem, while being solvable by such transformations. solvable by such transformations. For example, consider the five point dice shape with width 2 and height 2.



Clearly, a downward contraction is legal in this example. But $d_1 = \frac{\Delta_1}{2} - \Theta = \frac{2}{2} - 1 = 0$, and because no points lie on the locations q_1 and q_2, X_1 and X_2 are empty.
Later, the above conditions will in some special cases be weakened.
Without loss of generality, assume that the direction of the contraction is downward ($dir = \downarrow$), and p is the upper carrier.

Because $pl(T') > 0$, T' contains at least one line segment s on p, with $\mid s \mid > 0$. Let l be the leftmost point of s and r the rightmost point. Line segment s contains one or more downward connections, which will be referred to as dcs_1, dcs_2,..., dcs_μ, enumerated from left to right.
Proof of the Complex Contraction Theorem.
Suppose T' is a RMST. Then $l, r \in V \cap p$, so s contains at least $p_i \bullet\!\!-\!\!\bullet p_{i+1}$, where $p_i = l$. (Claims 1,2)

Claim 1 *$degree(l), degree(r) \notin \{3, 4\}$.*

Proof.
Trivial from the assumption that p is the upper carrier, $l, r \in p$ and that l, r are the outermost points of s. □

Claim 2 *If $l, r \notin V \cap p$, then $degree(l), degree(r) \notin \{1, 2\}$.*

Proof.
Suppose $degree(l) = 1$ (analogously for r). Then a positive length of line connects l to the right without downward connections. Because $l \notin V$, this line segment is obsolete.
Suppose $degree(l) = 2$ (analogously for r). As l is the leftmost point of s and $l \notin V$, l is a corner point and the L-shape with l as corner can be reversed. Contradiction to $CF(T', dir)$. □

The cases that remain to be considered are the ones where p consists only of (possibly several disjoint sets of) adjacent line segments between points in p. Consider such a collection of adjacent line segments $l = p_i \bullet\!\!-\!\!\bullet p_{i+1} \bullet\!\!-\!\!\bullet \cdots p_{j-1} \bullet\!\!-\!\!\bullet p_j = r$. From the above, it can be assumed that each of these line segments supports at most one downward connection.
If $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ contains no downward connections, $p_i \bullet\!\!-\!\!\bullet q_i' \bullet\!\!\leadsto\!\!\bullet x_i$ can be constructed at a cost $|p_i \bullet\!\!-\!\!\bullet q_i' \bullet\!\!\leadsto\!\!\bullet x_i| \leqslant d_i + \Theta \leqslant \frac{\Delta_i}{2}$. The resulting cycle can then be broken by deleting $p_i \bullet\!\!-\!\!\bullet p_{i+1}$, where $|p_i \bullet\!\!-\!\!\bullet p_{i+1}| = \Delta_i$. Therefore, $\mid T \mid < \mid T' \mid$, so T' is not a RMST. Contradiction. This transformation is illustrated in figure 2.

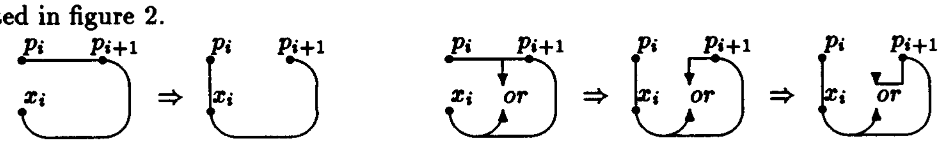


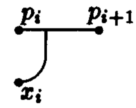
Figure 2: No downward connection or a rightmost downward connection

If $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ contains a downward connection dc_i where $dc_{i_x} > \frac{p_{i_x} + p_{i+1_x}}{2}$, $p_i \bullet\!\!-\!\!\bullet q_i' \bullet\!\!\leadsto\!\!\bullet x_i$ can be constructed at a cost $|p_i \bullet\!\!-\!\!\bullet q_i' \bullet\!\!\leadsto\!\!\bullet x_i \mid \leqslant d_i + \Theta \leqslant \frac{\Delta_i}{2}$. The resulting cycle can then be broken by deleting

$p_i \bullet\!-\!\bullet dc_i$, where $| p_i \bullet\!-\!\bullet dc_i | > \frac{\Delta_i}{2}$. Therefore, $| T | < | T' |$. Contradiction. This transformation is illustrated in figure 2.
The only remaining possibility is the one where $p_i \bullet\!-\!\bullet p_{i+1}$contains exactly one downward connection, dc_i, and $dc_{i_x} \leqslant \frac{p_{i_x}+p_{i+1_x}}{2}$.
Let dc'_i denote the projection in the direction dir of dc_i on q.

Claim 3 $dc_i \bullet\!-\!\bullet dc'_i \sqsubset x_i \bullet\!\sim\!\bullet p_i$.

Proof. Suppose $dc_i \bullet\!-\!\bullet dc'_i \not\sqsubset x_i \bullet\!\sim\!\bullet p_i$. Then $p_i \bullet\!-\!\bullet q'_i \bullet\!\sim\!\bullet x_i$ can be constructed at a cost $|p_i \bullet\!-\!\bullet q'_i \bullet\!\sim\!\bullet x_i| \leqslant d_i + \Theta \leqslant \frac{\Delta_i}{2}$. The resulting cycle can then be broken by deleting $dc_i \bullet\!-\!\bullet p_{i+1}$, where $| dc_i \bullet\!-\!\bullet p_{i+1} | \geqslant \frac{\Delta_i}{2}$.



Reversing the remaining L-shape produces an overlap of length Θ. Therefore, $| T | < | T' |$. Contradiction. This transformation is illustrated in figure 3.
□

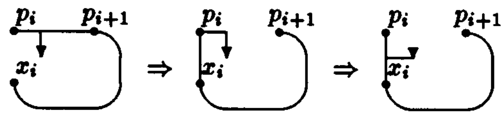


Figure 3: $dc_i \bullet\!-\!\bullet dc'_i \sqsubset x_i \bullet\!\sim\!\bullet p_i$

Claim 4 $\forall_{k \in \{i,\cdots,j-1\}} dc_k \bullet\!-\!\bullet dc'_k \sqsubset x_k \bullet\!\sim\!\bullet p_k$, *and* $dc_{k_x} \leqslant \frac{p_{k_x}+p_{k+1_x}}{2}$.

Proof. The claim holds for $k' = i$. Suppose that it holds for $k' \in \{i, \cdots, k-1\}$. Then it must be shown that it holds for $k' = k$ as well. Therefore, assume $\forall_{k' \in \{i,\cdots,k-1\}} dc_{k'} \bullet\!-\!\bullet dc'_{k'} \sqsubset x_{k'} \bullet\!\sim\!\bullet p_{k'}$ and $dc_{k'_x} \leqslant \frac{p_{k'_x}+p_{k'+1_x}}{2}$.

The proof of this claim will be subdivided as follows:

- $p_{k+1} \sqsubset x_k \bullet\!\sim\!\bullet p_k$
  - $p_k \bullet\!-\!\bullet p_{k+1}$ has no downward connections
  - $p_k \bullet\!-\!\bullet p_{k+1}$ has one downward connection
    - $dc_{k_x} > \frac{p_{k_x}+p_{k+1_x}}{2}$
    - $dc_{k_x} \leqslant \frac{p_{k_x}+p_{k+1_x}}{2}$
- $p_{k+1} \not\sqsubset x_k \bullet\!\sim\!\bullet p_k$
  - $dc_{k-1} \bullet\!-\!\bullet p_k \sqsubset x_k \bullet\!\sim\!\bullet p_k$
  - $dc_{k-1} \bullet\!-\!\bullet p_k \not\sqsubset x_k \bullet\!\sim\!\bullet p_k$
    - $p_k \bullet\!-\!\bullet p_{k+1}$ has no downward connections
    - $p_k \bullet\!-\!\bullet p_{k+1}$ has one downward connection
      - $dc_{k_x} > \frac{p_{k_x}+p_{k+1_x}}{2}$
      - $dc_{k_x} \leqslant \frac{p_{k_x}+p_{k+1_x}}{2}$

If $p_{k+1} \sqsubset x_k \bullet\!\sim\!\bullet p_k$ and $p_k \bullet\!-\!\bullet p_{k+1}$ has no downward connections, add $p_k \bullet\!-\!\bullet q'_k \bullet\!\sim\!\bullet x_k$ and delete $p_k \bullet\!-\!\bullet p_{k+1}$. Because $|p_k \bullet\!-\!\bullet q'_k \bullet\!\sim\!\bullet x_k| \leqslant d_k + \Theta \leqslant \frac{\Delta_k}{2} < \Delta_k = | p_k \bullet\!-\!\bullet p_{k+1} |$, $| T | < | T' |$, contradiction. This transformation is the same as in figure 2.

If $p_{k+1} \sqsubset x_k \bullet\!\sim\!\bullet p_k$ and $dc_{k_x} > \frac{p_{k_x}+p_{k+1_x}}{2}$, add $p_k \bullet\!-\!\bullet q'_k \bullet\!\sim\!\bullet x_k$ and delete $p_k \bullet\!-\!\bullet dc_k$. Because $|p_k \bullet\!-\!\bullet q'_k \bullet\!\sim\!\bullet x_k| \leqslant d_k + \Theta \leqslant \frac{\Delta_k}{2} < | p_k \bullet\!-\!\bullet dc_k |$, $| T | < | T' |$, contradiction. This transformation is analogous to the previous.

If $p_{k+1} \sqsubset x_k \bullet\!\sim\!\bullet p_k$ and $dc_{k_x} \leqslant \frac{p_{k_x}+p_{k+1_x}}{2}$, add $p_k \bullet\!-\!\bullet q'_k \bullet\!\sim\!\bullet x_k$, delete $dc_k \bullet\!-\!\bullet p_{k+1}$ and reverse the L-shape with dc_k as its corner, creating an overlap of Θ. Because $|p_k \bullet\!-\!\bullet q'_k \bullet\!\sim\!\bullet x_k| \leqslant d_k + \Theta \leqslant \frac{\Delta_k}{2} \leqslant | dc_k \bullet\!-\!\bullet p_{k+1} |$, $| T | + \Theta \leqslant | T' |$, contradiction. This transformation is the same as in figure 3.

Hence, one can assume that $p_{k+1} \not\sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$.

If $dc_{k-1} \bullet\!\!-\!\!\bullet p_k \sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$, add $p_k \bullet\!\!-\!\!\bullet q'_k \bullet\!\!\sim\!\!\bullet x_k$ and delete $dc_{k-1} \bullet\!\!-\!\!\bullet p_k$. Because $|p_k \bullet\!\!-\!\!\bullet q'_k \bullet\!\!\sim\!\!\bullet x_k| \leqslant d_k + \Theta \leqslant \frac{\Delta_k}{2} \leqslant |\ dc_{k-1} \bullet\!\!-\!\!\bullet p_k\ |$, $|\ T\ | \leqslant |\ T'\ |$. This transformation is illustrated in figure 4. The only situation when the above transformation does not reduce $|\ T'\ |$ arises when $dc_{k-1_x} = \frac{p_{k-1_x}+p_{k_x}}{2}$. Assume this to be the case, for otherwise the reduction of $|\ T'\ |$ would contradict ⑧. When $dc_{k-1_x} = \frac{p_{k-1_x}+p_{k_x}}{2}$, add $p_{k-1} \bullet\!\!-\!\!\bullet q'_{k-1} \bullet\!\!\sim\!\!\bullet x_{k-1}$ and delete $p_{k-1} \bullet\!\!-\!\!\bullet dc_{k-1} \bullet\!\!-\!\!\bullet dc'_{k-1}$. Because $|p_{k-1} \bullet\!\!-\!\!\bullet q'_{k-1} \bullet\!\!\sim\!\!\bullet x_{k-1}| \leqslant d_{k-1} + \Theta \leqslant \frac{\Delta_{k-1}}{2} < \frac{\Delta_{k-1}}{2} + \Theta = |\ p_{k-1} \bullet\!\!-\!\!\bullet dc_{k-1} \bullet\!\!-\!\!\bullet dc'_{k-1}\ |$, $|\ T\ | < |\ T'\ |$, contradiction. This additional transformation is illustrated in figure 5.

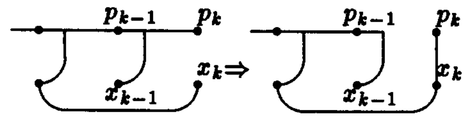


Figure 4: $dc_{k-1} \bullet\!\!-\!\!\bullet p_k \sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$

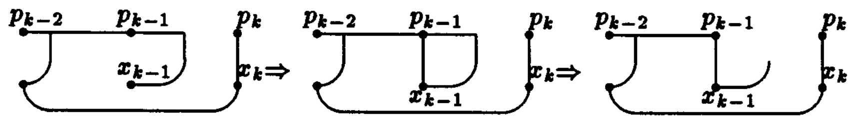


Figure 5: $dc_{k-1} \bullet\!\!-\!\!\bullet p_k \sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$ and $dc_{k-1_x} = \frac{p_{k-1_x}+p_{k_x}}{2}$

Hence, one can assume that $dc_{k-1} \bullet\!\!-\!\!\bullet p_k \not\sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$.
Furthermore, $p_k \bullet\!\!-\!\!\bullet p_{k+1}$ must possess downward connections, otherwise $p_{k+1} \sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$, contradicting the above.

If $dc_{k_x} > \frac{p_{k_x}+p_{k+1_x}}{2}$, add $p_k \bullet\!\!-\!\!\bullet q'_k \bullet\!\!\sim\!\!\bullet x_k$ and delete $p_k \bullet\!\!-\!\!\bullet dc_k$. Because $|p_k \bullet\!\!-\!\!\bullet q'_k \bullet\!\!\sim\!\!\bullet x_k| \leqslant d_k + \Theta \leqslant \frac{\Delta_k}{2} < |\ p_k \bullet\!\!-\!\!\bullet dc_k\ |$, $|\ T\ | < |\ T'\ |$, contradictoin. This transformation is illustrated in figure 6.

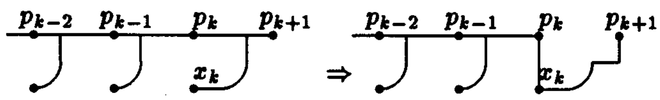


Figure 6: $p_{k+1} \not\sqsubset x_k \bullet\!\!\sim\!\!\bullet p_k$ and $dc_{k_x} > \frac{p_{k_x}+p_{k+1_x}}{2}$

The remaining possibility satisfies the claim. The correctness of the claim follows using natural induction.

□

From the above, it is clear that the line segment $p_i \bullet\!\!-\!\!\bullet p_j$ consists of line segments $p_k \bullet\!\!-\!\!\bullet p_{k+1}$ for $k \in \{i, \cdots, j-1\}$ which all possess a downward connection dc_k with $dc_{k_x} \leqslant \frac{p_{k_x}+p_{k+1_x}}{2}$, and $dc_k \bullet\!\!-\!\!\bullet dc'_k$ is part of $p_k \bullet\!\!-\!\!\bullet x_k$. The point x_j, however, must also be connected to $p_i \bullet\!\!-\!\!\bullet p_j$. It is obvious that the connection can not connect to p_j or use any second downward connection to some $p_k \bullet\!\!-\!\!\bullet p_{k+1}$, so x_k must connect to some $x_{k'}$, where $k' \in \{i, \cdots, j-1\}$. Now observe the line segment $p_{j-1} \bullet\!\!-\!\!\bullet p_j$. Using the above claim, $dc_{j-1_x} \leqslant \frac{p_{j-1_x}+p_{j_x}}{2}$.

If $dc_{j-1_x} < \frac{p_{j-1_x}+p_{j_x}}{2}$ then add $p_j \bullet\!\!-\!\!\bullet q'_j \bullet\!\!\sim\!\!\bullet x_j$ and delete $dc_{j-1} \bullet\!\!-\!\!\bullet p_j$. Because $|p_j \bullet\!\!-\!\!\bullet q'_j \bullet\!\!\sim\!\!\bullet x_j| \leqslant d_j + \Theta \leqslant \frac{\Delta_{j-1}}{2} < |\ dc_{j-1} \bullet\!\!-\!\!\bullet p_j\ |$, $|\ T\ | < |\ T'\ |$, contradiction. This transformation is illustrated in figure 7.

Otherwise, $dc_{j-1_x} = \frac{p_{j-1_x}+p_{j_x}}{2}$. Add $p_{j-1} \bullet\!\!-\!\!\bullet q'_{j-1} \bullet\!\!\sim\!\!\bullet x_{j-1}$ and $p_j \bullet\!\!-\!\!\bullet q'_j \bullet\!\!\sim\!\!\bullet x_j$ and delete $p_{j-1} \bullet\!\!-\!\!\bullet p_j$ and $dc_{j-1} \bullet\!\!-\!\!\bullet dc'_{j-1}$. Because $|p_{j-1} \bullet\!\!-\!\!\bullet q'_{j-1} \bullet\!\!\sim\!\!\bullet x_{j-1}| + |p_j \bullet\!\!-\!\!\bullet q'_j \bullet\!\!\sim\!\!\bullet x_j| \leqslant 2 \cdot (d_j + \Theta) \leqslant$

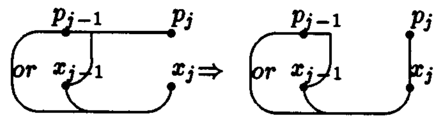


Figure 7: $dc_{j-1_x} > \frac{p_{j-1_x}+p_{j_x}}{2}$

$\Delta_{j-1} < \Delta_{j-1} + \Theta = | p_{j-1} \bullet\!\!-\!\!\bullet p_j | + | dc_{j-1} \bullet\!\!-\!\!\bullet dc'_{j-1} |$, $| T | < | T' |$, contradiction. This transformation is illustrated in figure 8.

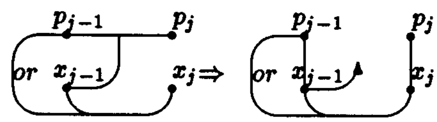


Figure 8: $dc_{j-1_x} = \frac{p_{j-1_x}+p_{j_x}}{2}$

This concludes the proof of the Complex Contraction Theorem.
□

Thus, no Rectilinear Minimal Steiner Tree for a set V satisfying the conditions of the contraction theorem for a direction *dir* and an outermost carrier p need contain a line segment $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ for any $i \in \{1..\pi - 1\}$. Hence, all points p_i can be *leaves* of T.

Recall that for all $i \in \{1..\pi\}$ the lines l_i are defined by $p_i \bullet\!\!-\!\!\bullet q'_i$, and that L is defined by $\cup_{i \in \{1..\pi\}} l_i$. Then, a Rectilinear Minimal Steiner Tree T for a set V satisfying the conditions of the Complex Contraction Theorem for a direction *dir* and a outermost carrier p can be constructed by $T := T' \cup L$, where T' is a RMST for $V' = V - p \cup q'$, and L is the set of line segments formed during the contraction.

The above immediately justifies the use of simple (one-point) contractions, for if $p = \{p_1\}$, $d_0 = d_\pi = \infty$ implies that the condition $(\forall_{i \in \{1 \cdots \pi\}}\ X_i \neq \emptyset) \wedge q \neq \emptyset$ is always true, and the above construction can be used to show that p_1 can always be contracted downward.
□

2.3.1 Consequences of the Complex Contraction Theorem

As was shown above, contractions are always legal if $\pi = 1$. So there is a RMST T which contains the line $p_1 \bullet\!\!-\!\!\bullet q'_1$. Because $p_1 \bullet\!\!-\!\!\bullet q'_1$ serves as a connection of p_1 to the rest of V only, $T' = T - p_1 \bullet\!\!-\!\!\bullet q'_1$ is a RMST for $V' = V - p_1 \cup q'_1$. This can be generalized for larger π. Therefore a contraction can be regarded as a transformation $\tau : \mathbf{R}^2 \mapsto \mathbf{R}^2$ and can be defined by $V \mapsto V - p \cup q'$. As was shown earlier, this transformation, when satisfying the conditions of the Complex Contraction Theorem, preserves minimality: if $| T |= t$ and $| T' |= t'$, then $t - t' = \Theta\pi$.

Definition 5 *A set V is* Complex Contractable *if and only if use of the Complex Contraction theorem applies and its application produces a Complex Contractable set V' or a set V' where* $\#V' = 1$.

Definition 6 (legal(R,dir)) *A contraction R in a direction dir is* legal *if there is a RMST T' for V' such that for some RMST T for V: $T = T' \cup L$.*

Lemma 3 *legal(R,dir)*$\Leftarrow (\forall_{i \in \{1,..,\pi\}} X_i \neq \emptyset) \wedge q \neq \emptyset$

Proof.
Follows almost directly from the Complex Contraction Theorem: Situations not yet considered:

- $CF(T', dir)$ does not hold. Then consider $T" = CF(T', dir)$. As $| CF(T', dir) | \leqslant | T' |$, proceed with $T"$.
- $CF(T', dir)$, but $pl(T') = 0$, trivial.

□

2.4 Global contractability

2.4.1 Introduction

Notice that for $n < 4$, no set V with $\#V = n$ exists that has more than one point on each side, so V is simple contractable to V', where $\#V' = 1$.

Consider $n \geqslant 4$. The four point square evidently does not satisty the conditions of the Complex Contraction Theorem ($\Delta_1 = \Theta < 2\Theta$), but could be solved using a contraction nevertheless. Suppose T, the RMST for this four point square contains $p_1 \bullet\!\!-\!\!\bullet p_2$, then some vertical line segment must exist connecting $p_1 \bullet\!\!-\!\!\bullet p_2$ to $V - p$. This implies that in this case, the rule $d_i = \frac{\Delta_i}{2} - \Theta$ can be weakened to $d_i = \Delta_i - \Theta$. This is called a Global Contraction.

Definition 7 *A* Global Contraction *of a set V is a Complex Contraction of a carrier p of V containing exactly two points, where the definition of d_i in the rules that the Complex Contraction has to satisfy, has been modified to $d_i = \Delta_i - \Theta$.*

2.4.2 Correctness

Corollary 5 *For $n \leqslant 4$, all cases can be solved minimally, using Global Contractions.*

Using the same reasoning, the rule $d_i = \Delta_i - \Theta$ proves to be sufficient for $n = 5$, where, up to isomorphisms, four different shapes exist after applying simple contractions.

If Complex Contractions are applied to a set V, $\#V = 5$, a set V' is produced for which $\Re(V')$ is degenerate or has at least two points of V' on each of its sides. That means that four of the five points are on the border of this smallest enclosing rectangle. The fifth point could be on the border also or it is in the interior. This means that the configurations, as depicted in figure 9 are possible.

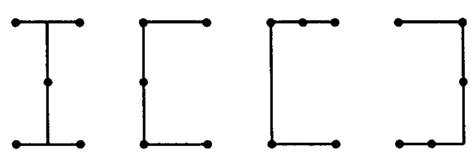


Figure 9: Possible five point shapes

Corollary 6 *For $n \leqslant 5$, all cases can be solved minimally, using Global Contractions.*

Definition 8 *A set V is* Global Contractable *if and only if use of the Global contraction corollary applies and its application produces a Global Contractable set V' or a set V' where $\#V' = 1$.*

Corollary 7 *Every set V with $\#V \leqslant 5$ is global contractable.*

2.5 General Contractions

A General Contraction is either a Global or a Complex contraction.

Theorem 2 (General Contraction Theorem)

$$legal(R, dir) \Leftarrow \begin{cases} (\forall_{i\in\{1\cdots\pi\}} X_i \neq \emptyset) \;\wedge\; q \neq \emptyset \;\; where \;\; d_i := \frac{\Delta_i}{2} - \Theta, \\ \qquad\qquad or \\ (\forall_{i\in\{1\cdots\pi\}} X_i \neq \emptyset) \;\wedge\; q \neq \emptyset \;\; where \;\; \#V \leqslant 5 \\ \qquad\qquad \wedge \;\; \pi = 2 \\ \qquad\qquad \wedge \;\; d_i := \Delta_i - \Theta \end{cases}$$

The correctness of this Theorem follows immediately from the correctness of Complex and Global Contractions.

Definition 9 *A set V is* General Contractable *if and only if use of the General Contraction Theorem applies and its application produces a General Contractable set V' or a set V' where* $\#V' = 1$.

3 The implementation of the contractions

3.1 The data structure

The structure containing the coordinates consists of grid elements, implemented as Pascal records with x and y coordinates, a boolean d and four pointers: up, rt, dn and lt. The illustrating figure also shows four additional linear lists, providing access to the rows and columns of points, earlier referred to as *carriers*, which are implemented as linear lists. This frame of access points simplifies the contraction procedure considerably, as will be shown below. The construction is straightforward, provided that the points are available sorted lexicographically on (x,y) as well as on (y,x)-tuple. Finally, the four linear lists are named xcoslo, ycoslo, xcoshi and ycoshi, meaning "the list which provides access to the rows of points, sorted on x-coordinate first in ascending order" etc.

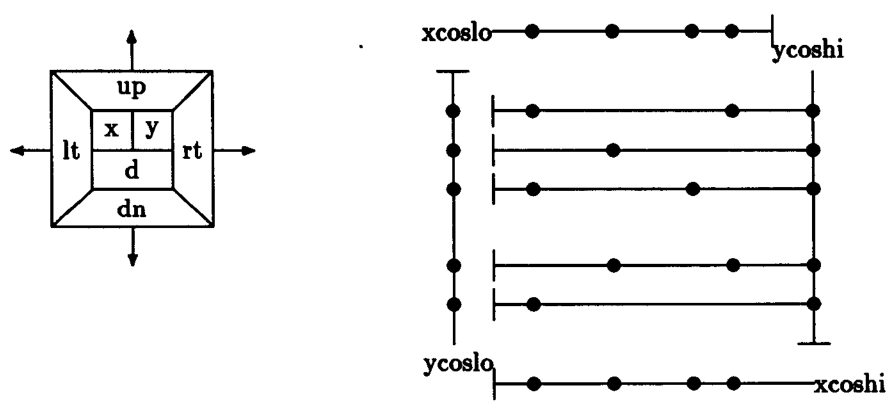


Figure 10: The data structure for *ycoshi*

3.2 Detecting a contractable side

The search for a contractable side proceeds counterclockwise around the set of points, until a round of four sides provides no further possibilities for contraction of any side. Without loss of generality, assume that the side considered is the upper one, which is checked for the possibility of a down-contraction. The first aspect to check is the *existence of a lower carrier*, which will be referred to as q. If this test fails, no contraction is possible and the procedure terminates. Proceeding along the upper carrier p, each pair of points (p_i, p_{i+1}) on p is checked for a point in X_i. During the walk over the p-carrier, all points of V must be checked for membership of the defined triangles. Because the triangles do not overlap, a simple linear algorithm performs this task, provided that the points are available ordered both horizontally and vertically.

If these conditions are met, the contraction can be performed. A cheap test revealing whether a side is *non*-contractable is possible [1]. The data structure is illustrated in figure 10.

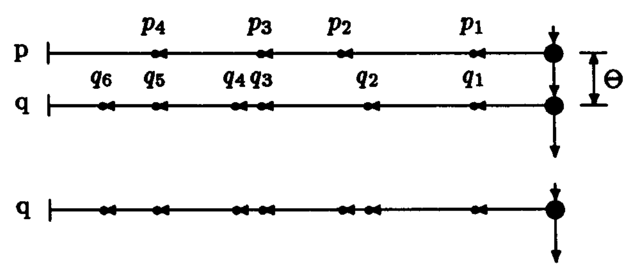


Figure 11: The merge step

Once the legality of a contraction is established, it can be performed by projecting p on q, that is, merging them. Because the data structure is, in essence, a linear list, the merging will not be discussed. The side-effects of the merge step, as shown in figure 11, are more interesting.

First, by the General Contraction Theorem, a point clashing with an other point during contraction can be discarded. This is easily done by short-circuiting the incoming and outgoing pointers and deleting the record that holds the point. More serious is the possibility that important global data may be disturbed, such as the points accessed by the access list. As rebuilding the data structure after every update takes too much time, disturbed data will be updated when found. The cost for this operation never exceeds (exactly) one step, for the access lists are used every round of contractions, so the cost of maintenance of the data is less than the cost of a merge.

Second, the side effects on the access lists themselves are only noticeable at the head or tail of the list. For example, the downward contraction in figure 4.2 causes the last element of ycoslo and the first element of ycoshi, as shown in figure 4.1, to be obsolete. Therefore the first element of ycoshi is deleted immediately and the last element of ycoslo is deleted when needed.

3.3 Contracting a side

Using the proposed structure, contract can easily be implemented by a standard merge. Consider the current example in figure 4.2. After establishing the validity of the down-contraction, proceed by taking the upper row and the one below it, merging them and replacing the lower one with the combination. Points with equal x-coordinates are deleted (in practice, one is marked by assigning *true* to its d(elete) field). The procedure terminates after cleaning up the now obsolete access points at the head of *ycoslo* and at the tail of *ycoshi*.

---

[1] Without loss of generality, assume that the direction of the contraction is downward. Let $b = p_{\pi_x} - p_{1_x}$. Simply check if $\sum_{i=1}^{\pi-1} p_{i+1_x} - p_{i_x} < 2(\pi - 1)\Theta$, or equivalently $b < 2(\pi - 1)\Theta$. So a contraction can be discarded immediately if $b < 2(\pi - 1)\Theta$. In case of a global contraction, the condition is $p_{2_x} - p_{1_x} < \Theta$.

3.4 Complexity

The implementation of the General Reduce Algorithm consists of five stages, numbered (I) through (V), which are discussed briefly. R denotes the number of contractions performed.

Algorithm	General Reduce		
Input	A set V of points with coordinates of the form (x, y)		
Output	A set V' of points		
begin	sort the points on lexicographically ascending (x, y)-tuple	(I):	$O(nlog(n))$
	sort the points on lexicographically ascending (y, x)-tuple	(I):	$O(nlog(n))$
	construct the four access lists and the grid structure	(II):	$O(n)$
	repeat for all sides do	(III):	$R\times$
	check if a lower carrier exists		$O(n)$
	check if all X_i triangles contain a point of V	(IV):	$O(n)$
	if these conditions are met then		
	$T := T \cup L$		$O(1)$
	merge the two carriers	(V):	$O(n)$
	update the access lists and the grid structure		$O(n)$
	until no contractable carrier exists		
end			

(I) $2.n.log(n)$: To build the proposed data structure, we need to sort the points of V twice, once on lexicographically ascending (x,y)–tuple, and once on lexicographically ascending (y,x)–tuple. (II) $4.n + n$: From the sorted data, the four access lists and the grid structure are constructed. (III) $2n$: Because there are at most n horizontal and vertical carriers, $R \leqslant 2n$ contractions can be performed. As described above R denotes the number of contractions performed. (IV) Because of the ordering of the non-overlapping triangles and the points, the implementation of the algorithm that establishes if every triangle contains at least one point is linear. Each contraction begins by checking which of the four sides is contractable. (V) n per contraction: Performing a contraction amounts to a merge of two linear lists.
Adding these results, the total complexity is

(I) $2.n.log(n)$
(II) $4.n + n$
(III) $R\times$
(IV) n
(V) n

It is clear that the time complexity of applying as many contractions as possible on a set V with $\#V = n$ is $O(n^2)$, and uses linear memory. More generally, performing R legal contractions costs $O(Rn)$ if $R = \Omega(logn)$ and $O(nlogn)$ otherwise.

4 A bound for the area covered by the RMST

Intuitively, a tree that possesses L-shapes pointing outward could benefit from reversion of these L-shapes. This would increase the probability of a connection to be made to the reversed L-shape, thereby decreasing the overall length of the tree. In this section, it will be shown that a RMST exists which does not extend beyond the boundary defined by its Rectilinear Internal Convex Hull (RICH). Intuitively, the RICH is the tightest contour of a set of objects possible, where all objects must be reachable using carriers. For example, the RICH of a set X of two points equals its enclosing rectangle, $\Re(X)$.

Definition 10

The partial RICH of a set V with respect to the direction dir $=\downarrow$, *denoted PRICH(dir,V), is defined as the boundary of the area* $\mathbf{R}^2$-Q, *where* Q *denotes the union of all left upper and right upper quadrants Q, with* $Q \cap V = \emptyset$. *The Rectilinear Internal Convex Hull of a set V of points in the plane is the boundary of the area enclosed by line segments of the partial RICH's of V*

Corollary 8
RICH(V)=line segments in PRICH($\uparrow$*,V)* $\cup$ *PRICH(*$\downarrow$*,V)* $\cup$ *PRICH(*$\leftarrow$*,V)* $\cup$ *PRICH(*$\rightarrow$*,V).*

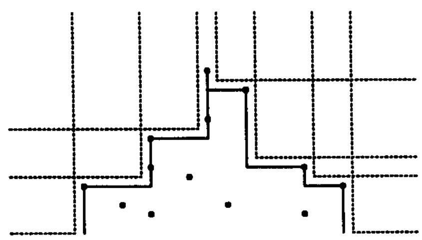


Figure 12: The construction of PRICH($\downarrow$,V)

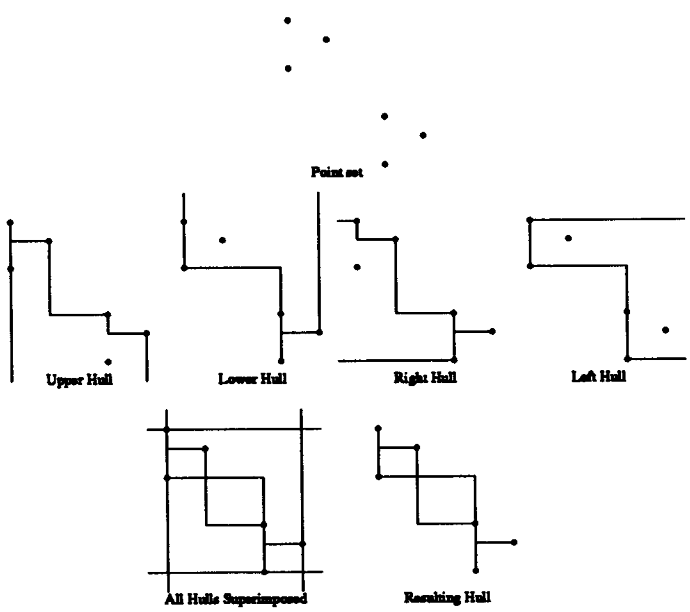


Figure 13: The construction of the RICH

It is easily seen that the Rectilinear Internal Convex Hull of a set of objects X as described here is equal to the RICH of the set of points in X together with the set of points that form the endpoints of the line segments in X.

The RICH for a set X of objects can be calculated in $O(nlogn)$ time by using a simple line sweep algorithm on the set V of points and endpoints of line segments in X, one in every direction. The event points are the elements of V. The line segments of the resulting partial hulls form the RICH.

The Rectilinear Inner Convex Hull of V will prove to be able to contain a RMST for V. Furthermore, any RMST for V can be transformed to a RMST for V within the boundary defined by RICH(V).

Theorem 3 (RICH) *Let T be a RMST for V. Let t be a structure used in T extending beyond RICH(V). Then T can be rebuilt to a tree T' with equal length that lies within the boundary defined by RICH(V).*

Proof. Let t be a connected structure in T" where T" is the forest of subtrees extending beyond RICH(V). Then t contains no vertices. Let a and b be the points of t that touch RICH(V) and have maximal distance to each other of all points of t that touch RICH(V). Then the tree t must have a length that is at least equal to the length of the path between a and b over the RICH, where this path is chosen to be the path $a \bullet\!\!\sim\!\!\bullet b$ where $t \cup a \bullet\!\!\sim\!\!\bullet b$ does not contain a cycle. But because t contains no vertices, it can safely be replaced by the path $a \bullet\!\!\sim\!\!\bullet b$, implying that a tree of shorter or equal total length exists that connects V.
□

Corollary 9 *If t is not an L-shape, then T is suboptimal.*

Corollary 10 *For all sets V of points in the Rectilinear plane, a RMST T exists that lies within RICH(V).*

Definition 11 *An articulation point $v \in V$ is a point where the boundaries of the RICH touch.*

Corollary 11 *For every articulation point $v \in V$, V can be separated into two sets of points V_1 and V_2, where $V_1 = V \cap Q_1$ and $V_2 = V \cap Q_2$ for two quadrants $Q_1 \neq Q_2$ that share v as their corner.*

Corollary 12 *For every articulation point $v \in V$ that divides V into two subsets V_1 and V_2 as described above, the RMST T for V can be constructed by joining T_1 and T_2, the RMSTs for V_1 and V_2, respectively.*

The construction used in the above corollary is depicted in figure 14.

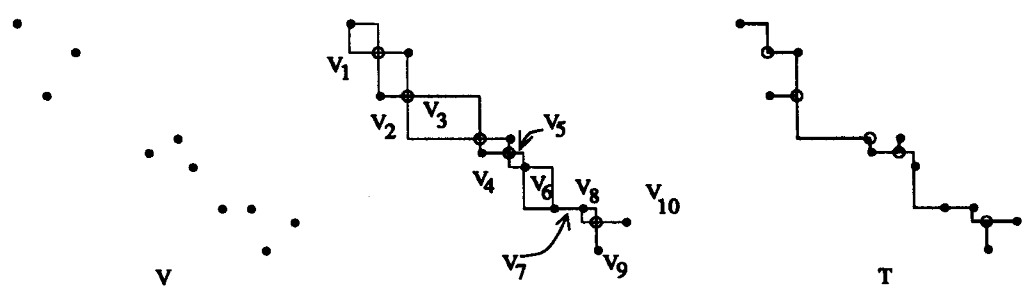


Figure 14: The construction T by dividing at articulation points

Corollary 13 *Simple contractions are legal.*

Proof. If a point is extreme in only one direction, it is easily seen that the RICH will enfold the point as tightly as possible. This implies that only one line segment within RICH can connect the point to the rest of the set. This is clear from figure 14. If a point is extreme in two directions, then the set V must possess an articulation point, and the RICH must possess a rectangle enclosing this articulation point and the extreme. The RMST is then separable at the articulation point into two subtrees T_1 and T_2, where T_1 is the RMST for the articulation point and the extreme. Then every optimal solution for the connection of the two points in T_1 will suffice, in particular both possible L-shapes. Therefore, a contraction in both directions in which the point is extreme is legal.
□

5 Approximations of RMSTs

As mentioned in the previous chapter, the length of a RMSpT for a set V is at most $\frac{3}{2}$ times larger than the length of the RMST for V[Hwang79].

An optimal algorithm to compute the Euclidean MSpT of for set V constructs a Voronoi Diagram for V in $O(nlogn)$ time, from which it derives a Delauney Triangulation. Thanks to the fact that the $O(n)$ edges of this triangulation suffice for the construction of the EMSpT, the total complexity of EMSpT is $O(nlogn)$. For a detailed discussion, the authors refer to [PrepSha85].

In [Hwang76], a similar construction is applied to obtain a Rectilinear MSpT in $O(nlogn)$ time. The implementation differs mainly in the separation of the points during recursion. More important is the difference in shape of the Rectilinear bisectors. The typical bisector of two points in the Rectilinear plane is not a straight line but a combination of two half-lines and a line segment, where the half-lines are parallel to either of the coordinate axes and the angle of the line segment with either the horizontal or vertical axis is 45°. The two half-lines are joined by a degenerate line segment of length 0 whenever the two points are on the same carrier. There is, however, a situation in which the bisector cannot be uniquely determined, or, alternatively, contains not only line segments but regions. This situation arises when the bisector is constructed for two points p_i and p_j where $| p_{i_x} - p_{j_x} | = | p_{i_y} - p_{j_y} |$. In this case, the bisector contains a slanted line segment and two quadrants. The vertical option can (arbitrarily) be chosen.

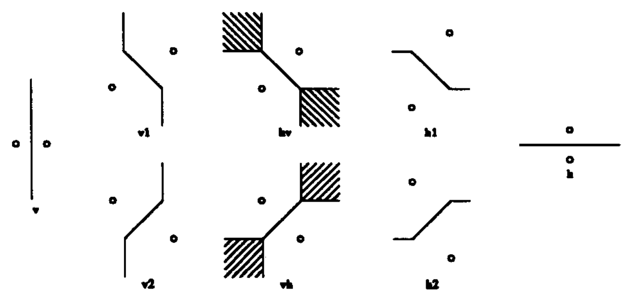


Figure 15: Possible Rectilinear bisectors

5.1 Generalized Voronoi Diagrams

In this section a method will be studied that reduces the suboptimality of an approximate RST by preventing the connection of a subtree to a vertex of another subtree whenever an edge is nearer. This method will prove to produce strictly better results than the standard RMSpT-based approach, although in contrived situations the difference in length of the approximations based on the two approaches may be arbitrarily small. A trivial algorithm can be used to implement these ideas. Instead of storing only the possible edges between points, it also stores those between points and edges. Initially, There are n subtrees that all contain exactly one point of V. The algorithm connects the nearest unconnected subtrees. The complexity is $O(n^2logn)$. It is easily seen that this problem subsumes the standard MSpT problem, thus implying at least $\Omega(nlogn)$ complexity. A far more elegant approach would use a generalized form of the Voronoi Diagram, where objects cannot only be points but also edges, that is, line segments. Furthermore, to facilitate construction of the approximation of the MST, a dynamic version of the algorithm is needed in order to insert newly constructed edges and virtual points. The Euclidean version of the problem has been solved and refined consecutively by [LeeDrys81], [?] and [Yap87]. Yap's article presents a conceptually simple method solving the problem for points, open line segments and open curve segments in optimal time $\Theta(nlogn)$, improving a factor $O(logn)$ over [LeeDrys81], where n represents the total number of objects in the object set X. The constructed Voronoi Diagram is a so-called Augmented Voronoi

Diagram, meaning that the locus of proximity of connected components (their combined Voronoi Regions) is subdivided into the loci of proximity of the constitutive objects. Yap's algorithm for the construction of the Euclidean (Augmented) Generalized Voronoi Diagram has a complexity of $O(nlogn)$.

Trivially, the bisector of a point and a line is a parabola. For example, the bisector of the point with Euclidean coordinates (0,a) and the x-axis with respect to a standard base is given by $\{(x,y) \mid y = \frac{x^2}{2a} + \frac{a}{2}\}$. The rectilinear bisector of the same objects consists of two line segments and two half-lines, defines as $\{(\pm a, y) \mid y \geqslant a\} \cup \{(x, \frac{a}{2} + \frac{|x|}{2} \mid x \in [-a, a]\}$.

Assumptions

The objects considered in the Rectilinear instance of the (A)GVD problem are points or open line segments parallel to either the horizontal or the vertical axis.

The bisector of a pair of points (p_1, p_2) and $(p_1 \pm r, p_2 \pm r)$ is chosen as in [Hwang76] (the vertical option).

Using the above assumptions, the construction used by Yap for the Euclidean instance can be used also in the Rectilinear case. Because the objects considered are line segments and points, the edges of the GVD are parabola- and line segments. The bisector of two (open) line segments can be determined in $O(1)$ time, and consists of up to seven segments of parabolas and lines. Therefore, the EGVD(X) where $\#X = n$ can not contain more than 21n-42 line- or parabola segments, and RGVD(X) can not contain more than 48n-96 line segments.

5.2 Dynamization

The procedure presented in the previous section offers a solution to the question of how to incorporate results of the Clustering theorems in an algorithm that approximates the Minimal Rectilinear Steiner Tree. In contrast to the Contraction theorems, that may discard the line segments used in an already connected set of points when performing another contraction, the connections resulting from Clustering must be taken into account before constructing other connections. In particular, the connections resulting from Clustering may offer possibilities for connection themselves. A contraction always produces line segments of which only the endpoint need be considered. In particular, no RMST possesses a connection to such a line segment using a Steiner point. Using this fact, an algorithm can perform contractions without paying any attention to line segments or L-shapes.

The connection produced by Clustering can be modeled by line segments or by line segments in combination with a corner point whenever the connection is a L-shape. Connections to corners of L-shapes can be modeled by inserting additional Steiner points. Applying the RAGVD algorithm to the set X consisting of these line segments, their Steiner or corner points and the points of V, an approximate RST A is produced that contains the connections resulting from the Clustering theorems.

Theorem 4 *Construct A' from the RMSpT M by inserting Steiner points and deleting overlapping line segments. The length of the RST A described above is at most equal to the length of A', implying $\frac{|A|}{|T|} \leqslant \frac{3}{2}$, where T denotes a RMST for V.*

Proof. This follows immediately from the Clustering theorems, which imply that the produced line segments or L-shapes create a cycle containing a line length or L-shape of greater length when inserted in A.

□

The RAGVD algorithm can be used to produce an approximation A that is at least as good as the RMSpT M. Probably, some edge will be connected to a line segment, Steiner point or corner point resulting from a Clustering. If this is the case, the resulting approximation will be strictly shorter than the RMSpT. In contrast to the construction of an approximation A' from the RMSpT, which may or may not prove to possess overlaps, this method ensures the existence of overlaps whenever a connection is made to an object from $X - V$. The authors emphasize that the approximation A' might be shorter that A, but A' depends on unpredictable construction sequences.

Corollary 14 *If a connection is made to a line segment or virtual point during the construction of the approximate RST A for V with RMST T, then* $\frac{|A|}{|T|} < \frac{3}{2}$.

The approach of connecting to line segments or virtual points resulting from Clustering theorems can be perfected by rebuilding the Voronoi Diagram whenever a new edge is determined. This approach diminishes the area of several Voronoi Regions whenever a line segment is inserted, causing a shorter connection to be used in the future if the distance of some object to the line segment will prove to be less than the distance that would have to be covered if the line segment had not been inserted.

The algorithm implementing the above method could call the RAGVD algorithm and build a partial approximation from it, and repeat this action until $n-1$ connections would be made. The complexity would then be $O(n^2 logn)$, mainly because of rebuilding parts of the Voronoi Diagram that have not changed at all. Fortunately, the various types of Voronoi Diagrams (EVD, RVD, EGVD, EAGVD, RGVD, RAGVD) all possess a property that allows updates to be made dynamically.

5.3 Order Decomposable Set Problems

In [Overmars81] a class of problems concerning sets of multidimensional objects is defined. A method of structuring sets such that the answer to a problem of this class can be maintained with low worst-case time bounds is given while insertions and deletions are performed. For the problem described in this section, the dynamization of Voronoi Diagrams to be used for the approximation of Rectilinear Minimal Steiner Trees, no objects need to be deleted, so the discussion will be restricted to updates after insertions only.

A set problem P is called $C(n)$-order decomposable if and only if there exist an ordering ORD and a function F such that for each set $X = \{x_1, \cdots, x_n\}$ ordered according to ORD and for all $i \in \{1, \cdots, n-1\}$ $P(x_1, \cdots, x_n) = \mathsf{F}\,(P(x_1, \cdots, x_i), P(x_{i+1}, \cdots, x_n))$, where F is a function that takes at most $C(n)$ time to compute when the set contains n elements and $C(O(n)) = O(C(n))$.

A set problem P is called order decomposable if and only if it is $C(n)$-order decomposable for some $C(n)$.

Let P be a $C(n)$-order decomposable set problem. There exists a divide-and-conquer solution to P that takes

- $O(n + ORD(n))$ steps when $C(n) = O(n^\epsilon)$ for $0 < \epsilon < 1$,
- $O(C(n) + ORD(n))$ steps when $C(n) = O(n^{1+\epsilon})$ for $\epsilon > 0$ and
- $O(logn \cdot C(n) + ORD(n))$ steps otherwise,

where n is the number of objects in the set and ORD is the time required to order n points according to ORD.

Given a $C(n)$-order decomposable set problem P, a dynamization procedure exists that allows an update time of $O(C(n))$ when $C(n) = \Omega(n^\epsilon)$ and $O(logn \cdot C(n))$ otherwise, where n denotes the current number of objects in the set.

From the above description, it is clear that the EVD, RVD, EGVD, RGVD, EAGVD and RAGVD problems are $O(n)$-decomposable, implying $O(n)$ update time. The corresponding dynamic versions of the problem will be refered to as Dynamic, denoted EDVD, RDVD, EDGVD, RDGVD, EDAGVD and RDAGVD, respectively.

Theorem 5 *An algorithm exists that constructs an approximation A of a RMST T for a set X of objects in the plane where X consists of V, a set of points, and open line segments and virtual points that result from the Clustering theorems, that has a worst-case complexity of* $O(n^2)$ *and* $\frac{|A|}{|T|} \leqslant \frac{3}{2}$.

Corollary 15 *If Contractions can be applied to V before applying the approximation algorithm, then the above result can be improved to* $\frac{|A|}{|T|} < \frac{3}{2}$ *while the complexity increases to* $O(R \cdot n)$ *if the number of contractions R exceeds logn.*

6 RSR < $\frac{3}{2}$

In Hwang's article 'On Steiner Minimal Trees with Rectilinear Distance' ([Hwang76]) the factor $\frac{|M|}{|T|} \leqslant \frac{3}{2}$ is proven. In this chapter a factor $\frac{|A|}{|T|} < \frac{3}{2}$will be proven, where A is not a RMSpT for V but an approximation for a RMST based on Voronoi, $| A | \leqslant | M |$.

The only sets for which the RMSpT M has length exactly $\frac{3}{2} | T |$ are the +-shaped sets.

6.1 Terminology and previous results

As in Hwang's article two operations on trees are defined: Shifting a line means moving a line, not containing a vertex, between two parallel lines until it is incident to a certain specified point. Reversing a L-shape means replacing it by the smallest enclosing rectangle of its endpoints minus the L-shape itself.

Figure 16: Shifting a line segment and reversing a L-shape

After shifts and reversions, the resultant graph is still a tree, and smaller or of the same length. Moreover the resultant tree is still a spanning tree (Steiner tree) if T was. If T' is obtained through shifts and reversions, T' is said to be equivalent to T. Let $\Xi(T)$ denote the set of all trees equivalent to the tree T. Let S be the set of all Rectilinear Minimal Steiner Trees. Partition S into $S_1 + S_2$ where $T \in S_1$ if and only if all vertices in T have degree one and $T \in S_2$ otherwise. Hwang proves $\frac{|M|}{|T|} \leqslant \frac{3}{2}$, by induction on the number of vertices. For $T \in S_2$, T can be split into two components at a vertex with degree two or more and apply induction on each component independently. Define $\Xi(S_2)$ as follows: $T \in \Xi(S_2)$ if a $T' \in \Xi(T)$ exists, such that $T' \in S_2$. The $\frac{3}{2}$-bound can be proved by working on T'. Hence $T \notin \Xi(S_2)$ is the only case which requires a proof. In §3 of Hwang's article it is shown that for any $T \notin \Xi(S_2)$, the induced subgraph of its Steiner points is a chain, called a Steiner chain. Define a staircase to be a continuous path of alternating vertical lines and horizontal lines such that their projections on the vertical and horizontal axes have no overlapping intervals.

Lemma 4 *(Hwang) Suppose* $T \notin \Xi(S_2)$*. Then:*
- *the Steiner chain is a staircase.*
- *If the number of Steiner points is greater than two, then either every vertical line segment (on the Steiner chain) contains more than one Steiner point (except perhaps the first and the last) and every horizontal line contains exactly one Steiner point, or vice versa.*

Since the problem is not affected by a 90° rotation, assume that if $T \notin \Xi(S_2)$, then the Steiner chain consists of a set of vertical lines where adjacent vertical lines are connected through a corner. Label the i^{th} Steiner point on the chain counting from above by ς_i.

Lemma 5 *(Hwang) Suppose* $T \notin \Xi(S_2)$*. Then every Steiner point must have a horizontal vertex line.*

Let γ be the number of Steiner points. If $T \notin \Xi(S_2)$, then T belongs to one of the following three types.

1. $\gamma = 1$: T is one of the trees in figure 17 with ς as the only Steiner point.

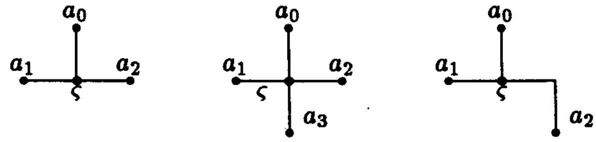


Figure 17: T has only one Steiner point.

2. $\gamma > 1$ and the Steiner chain is a straight line: the horizontal vertex lines at the sequence of Steiner points must alternate in the left-right direction (if ς_1 (ς_γ) has a corner line, we assume that ς_1 is the bottom point of a vertical edge (ς_γ is the top point of a vertical edge). Hence each Steiner point has exactly one horizontal vertex line. See figure 18.

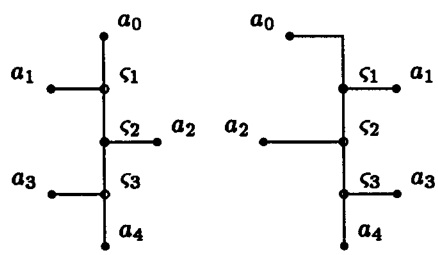


Figure 18: T has a straight Steiner chain.

3. γ is even and the Steiner chain is a straight line except the last two Steiner points are connected by a corner: without loss of generality, assume that ς_γ has a horizontal line segment to its left. Then each Steiner point except ς_γ has exactly one horizontal line segment which is a vertex line. These vertex lines alternate on the left-right direction. ς_1 and ς_γ both have vertical edges which are vertex lines. See figure 19.

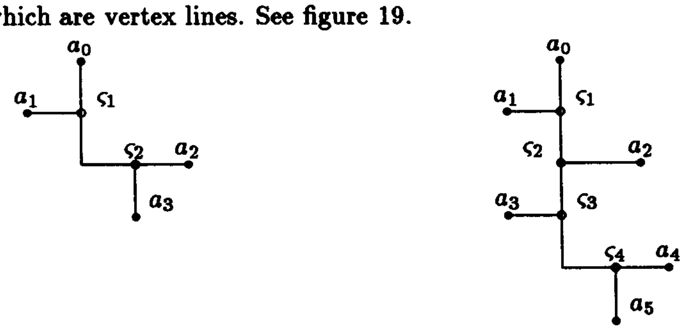


Figure 19: T has a straight Steiner chain with a corner at the end.

Lemma 6 *(Hwang) Suppose $T \notin \Xi(S_2)$ and $\#V = n$. Then γ, the number of Steiner points, is $n-2$ except when $n = 4$. Then γ could be either 1 or 2.*

For T belonging to type (2) or type (3) defined above, let h_i be the length of the horizontal vertex line at ς_i and let a_i be the vertex on it. If the vertical edge upward from ς_1 (vertical edge downward from ς_{n-2}) is the leg of a L-shape, then let h_0 (h_{n-1}) be the length of the horizontal leg of this L-shape. Otherwise let h_0 (h_{n-1}) have length zero. Define v_1 as the length of the vertical edge upward from ς_1, $v_i = \langle \varsigma_{i-1}, \varsigma_i \rangle$ for $2 \leqslant i \leqslant n-2$, and v_{n-2} as the length of the vertical edge downward from ς_{n-2}.

6.2 An approximation A, with $\frac{|A|}{|T|} < \frac{3}{2}$

In this subsection A is a RST, denoting an approximation for a RMST for a set V. Let M denote a RMSpT for V. A is constructed in three steps:

1. apply all possible contractions on the set V yielding a set of line segments L and a set V'
2. construct the RICH for V'
3. make an approximation A' for V' using RDAGVD, with the restriction that all edges lie within the RICH for V'.
4. $A := L \cup A'$

Theorem 6 *(Hwang)* $\frac{|M|}{|T|} \leqslant \frac{3}{2}$*, where T is a RMST for V, and M a RMSpT for V.*

Corollary 16 $\frac{|A|}{|T|} \leqslant \frac{3}{2}$

Corollary 17 *If A contains a Steiner point then* $\frac{|A|}{|T|} < \frac{3}{2}$

Proof
Suppose A contains a Steiner point ς. Then A also contains an overlap of length $\varepsilon > 0$, where ε is the length of the shortest line segment connected to ς. Because $| M | \geqslant | A | + \varepsilon$,

$$\frac{| A |}{| T |} \leqslant \frac{| M | - \varepsilon}{| T |} < \frac{| M |}{| T |} \leqslant \frac{3}{2}$$

□

Lemma 7 *If $R \geqslant 1$, then* $\frac{|A|}{|T|} < \frac{3}{2}$.

Proof
Let T' be a RMST for V' and T a RMST for V such that $T = T' \cup L$. Then $\frac{2}{3} \mid M' \mid \leqslant \mid T' \mid$ (Hwang), which implies that $\frac{2}{3} \mid A \mid \leqslant \frac{2}{3}(\mid M' \mid + \mid L \mid) = \frac{2}{3} \mid M' \mid + \frac{2}{3} \mid L \mid < \mid T' \mid + \mid L \mid = \mid T \mid$. □

Corollary 18 $\forall T \notin \Xi(S_2) : \frac{|A|}{|T|} < \frac{3}{2}$.

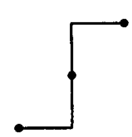
Proof
For all $T \notin \Xi(S_2)$ holds that at least two simple contractions are possible, at a_1 and a_n. □

Corollary 19 $\#V \leqslant 2 \Rightarrow \frac{|A|}{|T|} = \frac{|M|}{|T|} = 1$

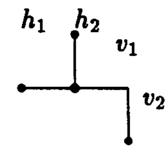
Lemma 8 $\#V = 3 \Rightarrow \frac{|M|}{|T|} \leqslant \frac{4}{3}$

Proof

Suppose two points of V are extremes in two directions, let M be a RMSpT for this set V. Then there is a RMST T for this set V such that $M = T$, satisfying $\frac{|M|}{|T|} = 1 \leqslant \frac{4}{3}$.



So suppose only one point of V is a double-extreme. Then any RMSpT can be routed over the rectangle. Without loss of generality assume that the three points are as depicted in the adjacent figure. Then $|T| = h_1 + h_2 + v_1 + v_2$ and $|M| = 2|T| - |corner|$, where *corner* is the length of the longest L-shape. Hence $|M| \leqslant \frac{2}{3}(2|T|)$, so $|M| \leqslant \frac{4}{3}|T|$. This concludes the proof of the lemma.
□



Lemma 9 *If $T \notin \Xi(S_2)$ is a RMST for a set V, $\#V = 4$, and T has two Steiner points connected by a L-shape, then $|M| < \frac{3}{2}|T|$, where M is a RMSpT for V.*

Proof

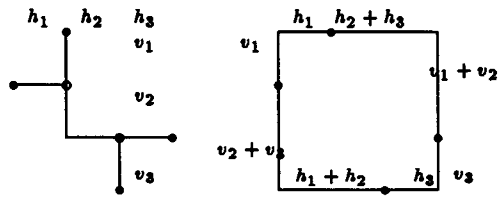


Figure 20: RMST for four vertices, the two Steiner points are connected by an L-shape

The situation has been depicted in figure 20. The worst RMSpT is found when routed over the boundary of $\Re(T)$. There are four possibilities to create a RMSpT:

• The upper-left corner is not part of the RMSpT M
This implies that $|M| = h_1 + 2h_2 + 2h_3 + v_1 + 2v_2 + 2v_3$
and $v_1 + h_1 \geqslant h_2 + h_3 + v_1 + v_2 \Leftrightarrow h_1 \geqslant h_2 + h_3 + v_2$
and $v_1 + h_1 \geqslant v_3 + h_3$
and $v_1 + h_1 \geqslant h_1 + h_2 + v_2 + v_3 \Leftrightarrow v_1 \geqslant h_2 + v_2 + v_3$

Suppose $\frac{2}{3}|M| = |T|$, this is true if and only if $\frac{2}{3}(h_1 + 2h_2 + 2h_3 + v_1 + 2v_2 + 2v_3) = h_1 + h_2 + h_3 + v_1 + v_2 + v_3 \Leftrightarrow h_2 + h_3 + v_2 + v_3 = h_1 + v_1 \geqslant h_2 + h_3 + v_2 + h_2 + v_2 + v_3 \Leftrightarrow 0 \geqslant h_2 + v_2$ which would mean there is only one Steiner point, contradiction.
• The upper-right corner is not part of the RMSpT M.
• The lower-right corner is not part of the RMSpT M.
• The lower-left corner is not part of the RMSpT M.
These last three cases can be solved in a similar way. □

Theorem 7 (Bound) $\frac{|A|}{|T|} < \frac{3}{2}$

Proof
Not using the knowledge that all edges of A lie within the RICH for A, the following can be proved.

1. If $T \notin \Xi(S_2)$ then $\frac{2}{3}|A| < |T|$.
2. If $T \in \Xi(S_2)$ then if at some point during the induction a subtree T' of T, which will not be split again, is cut loose for which holds:
   - $\#(T' \cap V) \leqslant 3$ or
   - $\#(T' \cap V) > 4$ or
   - $\#(T' \cap V) = 4$ but T' has the configuration of figure 20 or of figure 24

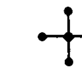
   Then $\frac{2}{3}|A| < |T|$

3. Otherwise $T \in \Xi(S_2)$ and all subtrees T' of T, which will not be split again, have a plus-shape as in figure 25: one Steiner point, which is of degree four, all four points have equal distance to each other, as will be proven in lemma 10.

By induction on $\#V = n$.
In this proof regular use is made of M, a RMSpT instead of A, the approximation. This use is justified because $|A| \leqslant |M|$ as discussed above. This theorem is trivially true for $n = 2$. Because of lemma 6.2 the theorem holds for $n = 3$. Suppose the theorem holds for all V' with $\#V' < n$. Consider a set V with $\#V = n$. This theorem is true for all V such that a RMST T for V is not in $\Xi(S_2)$. So let $T' \in \Xi(S_2)$. Then T' has a vertex x with degree two or more. We can split T' at x into two components, both containing x, and apply induction on each.

Now it could be that one of the components is a RMST T such that $T \notin \Xi(S_2)$. Unfortunately, corollary 18 can not be applied, for the subgraph T could lie entirely within the original RMST T', implying that no contraction could be applied on the subset of vertices lying in T. And no part of the RICH for T is part of the RICH for T'. From lemma 6.2, assume $n > 3$.

If $n = 4$ and T has only one Steiner point, as shown in the adjacent picture, a RMSpT for this set has the $\frac{2}{3}$-bound. This case will be treated later.



The strategy is to partition T at a Steiner point, say ς_q, into two subgraphs T_1 and T_2 such that T_1 is the induced subgraph of $\{a_0, a_1, ..., a_{q-1}\}$ plus the edge between ς_{q-1} and ς_q, and T_2 is the induced subgraph of $\{a_q, a_{q+1}, ..., a_{n-1}\}$.

In the following h_q cannot be added to the length of T_1. If h_q was added to T_1 and to T_2, then h_q would be counted twice, therefore $\frac{2}{3} \mid M \mid < \mid T_1 \cup T_2 \mid$ does not necessarily hold. However, observe that for the special choice of q that T_2 consists of only one vertex and $h_q > 0$, $T_1 \cup T_2 \neq T$ holds and hence $|T_1| + |T_2| = |T_1 \cup T_2| < |T|$.

A path p_1 on the set of points $\{a_0, a_1, ..., a_q\}$ will be constructed such that

$$\frac{2}{3} \mid p_1 \mid < \mid T_1 \mid$$

Let M_2 be the RMSpT on the set of points $\{a_q, a_{q+1}, ..., a_n\}$. Then $\frac{2}{3} \mid M_2 \mid < \mid T_2 \mid$ by the induction hypothesis. Hence p_1 and M_2 together are a spanning tree whose length is less than $\frac{3}{2}(|T_1| + |T_2|) \leqslant \frac{3}{2} |T|$. The theorem is then proved.

The selection of p_1 depends on whether there exists a T_1 such that the total length of its horizontal lines is sufficiently small in proportion to $|T_1|$. To be exact, suppose there exists a k, $1 \leqslant k \leqslant n-2$, such that

$$\frac{2}{3}\left(\sum_{i=0}^{k-1} h_i + \sum_{i=1}^{k} h_i + \sum_{i=1}^{k} v_i\right) < \sum_{i=0}^{k-1} h_i + \sum_{i=1}^{k} v_i.$$

Set $q = k$. Then p_1 can be selected as the path connecting $a_0, a_1, ..., a_k$ in that order. The length of p_1 is the bracket of the above equation, the length of T_1 is the right-hand side of the above equation. Clearly $\frac{2}{3} \mid p_1 \mid < \mid T_1 \mid$.
Therefore, now assume

$$\sum_{i=1}^{k} v_i \leqslant \sum_{i=1}^{k} h_i + (h_k - h_0), 1 \leqslant k \leqslant n-2.$$

A tree of type (2) or type (3) remains so if we reverse the ordering of its Steiner points (for type (3) tree, push the corner of the Steiner chain to the other end). Therefore the existence of a subscript j, $2 \leqslant j \leqslant [n/2] + 1$ can be assumed, such that $h_i > h_{i-2}$ for all $i = 1, ..., j-1$, and $h_j \leqslant h_{j-2}$. Set $q = j$. Connect $(a_j, a_{j-2}, a_{j-4}, ..., a_0, ..., a_{j-3}, a_{j-1}, a_j)$ in that order, i.e., connect all vertices on the same side in order before crossing to the other sides. This way a tour t is created on this set of $j+1$ vertices. The length of t equals the periphery (denoted by $|R|$) of the smallest enclosing rectangle of the $j+1$ vertices in t. The desired p_1 can be obtained by deleting a proper link in t. For all j $\frac{2}{3} \mid p_1 \mid < \mid T_1 \mid$ will be proved.

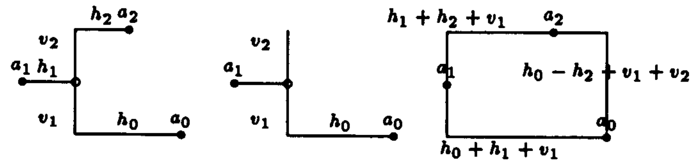


Figure 21: Part of a Steiner chain containing 3 vertices

If $j = 2$ then T_1 is as depicted in figure 21. The corner of maximum length must be deleted in t. Recall here that h_2 is not added to the cost of T_1. $|\, t \,| = 2(h_0 + h_1 + v_1 + v_2) = 2 \,|\, T_1 \,|$. Hence $|\, p_1 \,| \leqslant \frac{2}{3} \,|\, t \,| = \frac{4}{3} \,|\, T_1 \,|$. So if T_1 contains three or less vertices the theorem is proved.

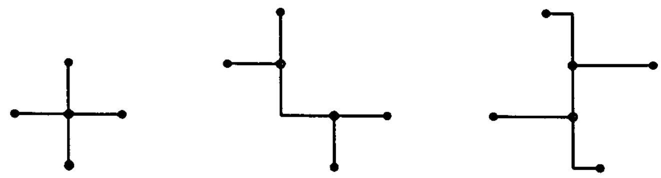


Figure 22: All possible RMST's T for four vertices, not $T \in \Xi(S_2)$

Suppose $j = 3$, then T_1 contains three vertices, two Steiner points and a loose edge, as shown in the last two pictures of figure 22. In T_1 the two Steiner points are connected by a L-shape or they are connected by a straight line segment. If the Steiner points are connected by a L-shape, then there are two possible configurations for T_1. If it is of the configuration in figure 20 then it has already been treated.

The other situation with four points and a L-shape between the two Steiner points is the one depicted in figure 23, the tree is the last part of a Steiner chain.

Figure 23: Part of a Steiner chain with a corner between the last two Steiner points

The worst RMSpT is found when routed over the boundary of $\Re(T)$. There are four possibilities to create a RMSpT: one of the corners is not part of the RMSpT p_1. For each of these cases a proof can be given, similar to that in lemma 9.

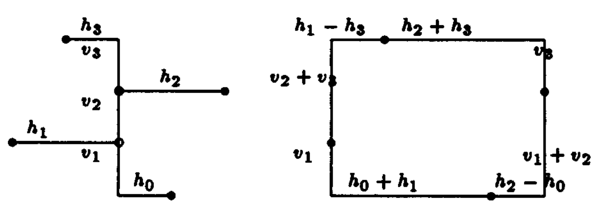
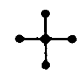


Figure 24: part of a Steiner chain containing 4 vertices

Suppose T_1 has two Steiner points, which are connected by a straight line segment, as in figure 24. Note that h_3 cannot be added to T_1 as it will be added to T_2. In this situation a RMSpT M_1 could exist such that $| M_1 | = \frac{3}{2} | T_1 |$. To prove the theorem correct it must be guaranteed that $| M_2 | < \frac{3}{2} | T_2 |$. The only situation giving rise to this problem is the one in figure 24, this situation cannot be repeated indefinitely. So in the end h_3 is the uppermost horizontal line segment in T. The last partition is then a tree T_2' containing a_3 for which the RMSpT M_2' consists only of a_3. In this case $T_1 \cup T_2 \neq T$, so $| T_1 \cup T_2 | < | T |$. This implies the theorem holds in this situation. Or do not perform this last partition and add h_3 to the remaining tree T_1'. Proof, similar to the proof of lemma 9, can be given of $| M_1' | < \frac{3}{2} | T_1' |$.

As discussed above h_3 was not added to T_1, so $| T_1 | = h_0 + h_1 + h_2 + v_1 + v_2 + v_3$. $| p_1 | \leqslant \frac{3}{2} | T_1 |$ for $| p_1 | \leqslant \frac{3}{4} | t | = 2(h_1 + h_2 + v_1 + v_2 + v_3) = \frac{3}{2}(| T_1 | - h_0)$. If $h_0 > 0$ then $| p_1 | < \frac{3}{2} | T_1 |$, but h_0 can be zero, so $| p_1 | \leqslant \frac{3}{2} | T_1 |$.

Suppose $j = 3$ and T_1 has only one Steiner point, as shown in the first picture in figure 22, then not $T \notin \Xi(S_2)$, contradiction. So the theorem is correct for $j = 3$.



Now consider $j \geqslant 4$. Let $\sum_{i=0}^{j-3} h_i = \Theta | R | \geqslant 0$. Then the four links $a_{j-4} \to a_{j-2} \to a_j \to a_{j-1} \to a_{j-3}$ in the tour t have a total length of

$$| R | - h_{j-3} - h_{j-4} - \sum_{i=1}^{j-4} v_i - \sum_{i=1}^{j-3} v_i$$

$$\geqslant | R | - h_{j-3} - h_{j-4} - \left[\sum_{i=1}^{j-4} h_i + (h_{j-4} - h_0)\right] - \left[\sum_{i=1}^{j-3} h_i + (h_{j-3} - h_0)\right]$$

$$= | R | - h_{j-3} - h_{j-4} - \sum_{i=1}^{j-4} h_i - h_{j-4} + h_0 - \sum_{i=1}^{j-3} h_i - h_{j-3} + h_0$$

$$= | R | - h_{j-3} - 2h_{j-4} + 2h_0 - 2\sum_{i=1}^{j-3} h_i$$

$$= | R | + 4h_0 - h_{j-3} - 2h_{j-4} - 2\sum_{i=0}^{j-3} h_i$$

Now suppose $j = 4$ then this computes

$$| R | - 2(h_0 + h_1) + 4h_0 - 2h_0 - h_1 = | R | - 3h_1$$

$$> | R | - 4\sum_{i=0}^{1} h_i = | R | (1 - 4\Theta).$$

So the theorem holds for $j = 4$. The last possibility is $j > 4$, then the above computes as

$$| R | - 2\sum_{i=0}^{j-3} h_i - 2h_{j-4} - h_{j-3} + 4h_0 >$$

$$| R | - 2\sum_{i=0}^{j-3} h_i - 2h_{j-4} - 2h_{j-3} - 2\sum_{i=0}^{j-5} h_i$$

$$= | R | - 3\sum_{i=0}^{j-3} h_i - h_{j-3} - h_{j-4} - \sum_{i=0}^{j-5} h_i$$

$$=| R | -4\sum_{i=0}^{j-3} h_i =| R | (1-4\Theta)$$

If the link of maximal length among these four links from t is deleted, a path p_1 of the $j+1$ vertices is obtained with length

$$<| R | -\frac{1}{4} | R | (1-4\Theta) =| R | \left(\frac{3}{4}+\Theta\right).$$

Multiplying this length by $\frac{2}{3}$ the following is obtained:

$$\frac{2}{3} | p_1 |<| R | \left(\frac{1}{2}+\frac{2}{3}\Theta\right).$$

But the length of T_1 is

$$\sum_{i=0}^{j-1} h_i + \sum_{i=1}^{j} v_i = \sum_{i=0}^{j-3} h_i + \frac{1}{2} | R |=| R | \left(\frac{1}{2}+\Theta\right)$$

Conclusion:

$$\frac{2}{3} | p_1 |<| R | \left(\frac{1}{2}+\frac{2}{3}\Theta\right) < \frac{1}{2} | R | +\Theta | R |=| R | (\frac{1}{2}+\Theta) =| T_1 | .$$

The remaining situation is item 3 in the glossary of the proof. First the statement in this item will be proved, see lemma 10. Then $\frac{|A|}{|T|} < \frac{3}{2}$will be proved for this situation.

Definition 12 *A diamond-shaped set of points* $\Diamond$ *is a set of four points such that there is a point* $p = (p_x, p_y)$ *and a real number* r*, the radius of the diamond-shape, for which* $\Diamond = \{(p_x+r, p_y), (p_x - r, p_y), (p_x, p_y + r), (p_x, p_y - r)\}$

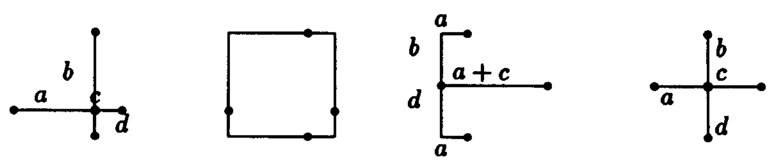


Figure 25: plus-shape

Lemma 10 *The only set of four points for which* $\frac{2}{3} | M |=| T |$ *is the diamond-shaped set.*

Definition 13 *The RMST T for a diamond-shaped set of four points* $\Diamond$ *consists of a Steiner point in the middle of the set and four line segments connecting each point to the Steiner point, and is called a +-shape.*

Proof

Note that contractions are applicable on this set, so $\frac{2}{3} | A |<| T |$. All configurations of four points, having 2 Steiner points, have been treated above. So consider all configurations of four points for which the RMST has one Steiner point. Hwang showed that this Steiner point must have two vertical and two horizontal vertex lines, as depicted in figure 25.

Suppose there is a RMSpT M for this configuration that is routed over the boundary of the smallest enclosing rectangle. Then there are four possibilities to create M, for there are three corners of this rectangle in M. Without loss of generality only one of these cases will be treated.

Suppose the upper-left corner of the rectangle is not used in M.

This implies that $| M |= a + b + 2c + 2d$ and

1. $a+b \geqslant b+c \Leftrightarrow a \geqslant c$

2. $a+b \geqslant c+d$

3. $a+b \geqslant a+d \Leftrightarrow b \geqslant d$

Moreover $\frac{2}{3} \mid M \mid = \mid T \mid$, implying that $c+d = a+b$. With equation 1 it follows that $d \geqslant b$ and hence, using equation 3, $b = d$. This leads to $a = c$.

So if there is a RMSpT, which is routed over the boundary of the rectangle, then the $\frac{2}{3}$-bound is only reached if all vertices have equal distances to each other.
The remaining case is that all RMSpT's have an edge through the interior of the rectangle, see figure 25. Without loss of generality suppose that this edge connects the vertices which have their horizontal carrier in common. This implies that $a+c$ is less than any of the corners of the rectangle and less than $b+d$. Without loss of generality assume that the upper-left and the lower-left corner of the rectangle are used in M as well. This implies $a < b$, $c < b$, $a < d$, $c < d$ and $a \leqslant c$.
Now suppose $\frac{2}{3} \mid M \mid = \mid T \mid$, so $\frac{2}{3}(3a+b+c+d) = a+b+c+d \Leftrightarrow 3a = b+c+d > a+a+a$, which is a contradiction.
This concludes the proof of the lemma.
□

If the knowledge that all edges of A lie within the RICH for A, is not used, then the only RMST's T for a set V for which $\frac{2}{3} \mid A \mid < \mid T \mid$ does not hold, are those trees, which will be split by the induction into subtrees with four points, where these four points have equal distance to each other. So $\#V = 3k+1$ for certain $k > 1$. The theorem would be proved if each of these trees T could be contracted at least once. But figure 27 shows an example of a RMST T which does not satisfy the conditions of the Complex Contraction Theorem and which consists of plus-shapes. To the left of this RMST T a RMSpT M is shown for the same set of vertices. This RMSpT consists of several possible RMSpT's for a single diamond-shape. All possible RMSpT's for a single diamond-shape have been depicted in figure 26.

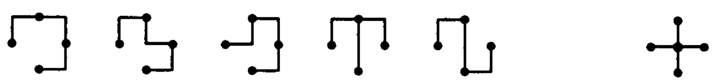


Figure 26: All possible RMST's for a *single* diamond-shape

The following shows that for the approximation A for a +-shaped set $V \mid A \mid < \frac{3}{2} \mid T \mid$ holds.

The trees A that could be produced where $\frac{|A|}{|T|} = \frac{3}{2}$ can be uniquely partitioned into diamond-shaped sets of four points where these sets touch at the corners. The algorithm ensures that no edge extends beyond the RICH of the set V. The behaviour of the edges can be described using the following notion:

Definition 14 *The frame* ⊞ *of a diamond-shaped set of points* ◊ *is the set containing twelve closed line segments on the carriers of* ◊, *bounded by the corners of the enclosing rectangle of* ◊, *the middle of* ◊ *and its points.*

As was shown earlier, the construction of a RST A containing a Steiner point proves the theorem, because then certainly $\frac{|A|}{|T|} < \frac{3}{2}$. Therefore, assume that it is possible to determine a construction sequence such that the RDAGVD algorithm produces no Steiner points when applied to a set where $\frac{|M|}{|T|} = \frac{3}{2}$ is described above.

The $3k$ connections in the tree covering $3k+1$ points in k diamond-shapes that have non-intersecting frames can be classified as follows:

1. connections that use one or more parts of frames while connecting points in different diamond-shapes,

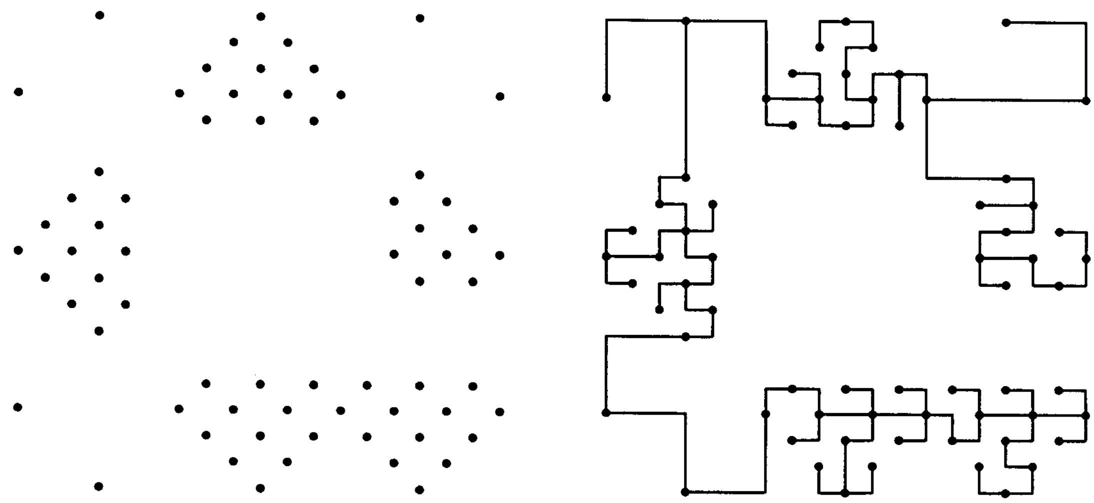


Figure 27: A non-contractable set that can be partitioned into diamonds

2. connections that use no parts of frames while connecting points in different diamond-shapes,
3. connections that use the inner frame segments of one diamond-shape,
4. connections that use the outer frame segments of one diamond-shape.

Type 3 connections can occur only when the algorithm will not use a shorter connection in the next step as a result of this construction. This can only be avoided when the point whose minimal distance is reduced is already connected to either of the points joined by the type 3 connection. Because only one of these connections is possible within the interior of a frame, at most k type 3 connections are possible in the tree.

Type 2 connections are only possible when the points joined are on empty sides of frames, where a side of a frame is called empty if no other diamond-shape shares the point. The illustrating figure 28 shows the two possibilities, and a third connection that uses an L-shape that partly covers a frame segment. Because this situation is subsumed by the one where no frame segment is overlapped (because there is no possibility for connection to the corner), it is classified as a type 2 connection. As is easily seen, the number of free sides of frames is equal to $2k + 2$, but at least four of these are outside the RICH, implying that at most $2k - 2$ free sides of frames can exist. Because c free sides of frames can cause at most $c - 1$ type 2 connections, at most $2k - 3$ type 2 connections are posssible in the tree.

Type 1 connections can only be constructed if the point whose minimal distance is reduced is already connected to either of the points joined by the type 1 connection. Because this is only the case when the type 1 connection can be replaced by a type 3 connection, and these connections are mutually exclusive, the total number of type 1 and 3 connections is k.

Type 4 connections can only exist if the two sides of frames covered are free. Therefore, at most $2k - 3$ type 2 and 4 connections are possible.

Summarizing, at most $k + 2k - 3 = 3k - 3$ of the $3k$ connections can be constructed by the RDAGVD algorithm without causing a Steiner point in the next step. Therefore, the RDAGVD algorithm can only produce RSTs A where $\frac{|A|}{|T|} < \frac{3}{2}$.
This concludes the proof of the Bound-theorem. □

7 Clustering

In section 2, the concept of contractions has been introduced. Contractions are useful in the sense that part of the RMST can be constructed in advance of making an approximation. Contractions

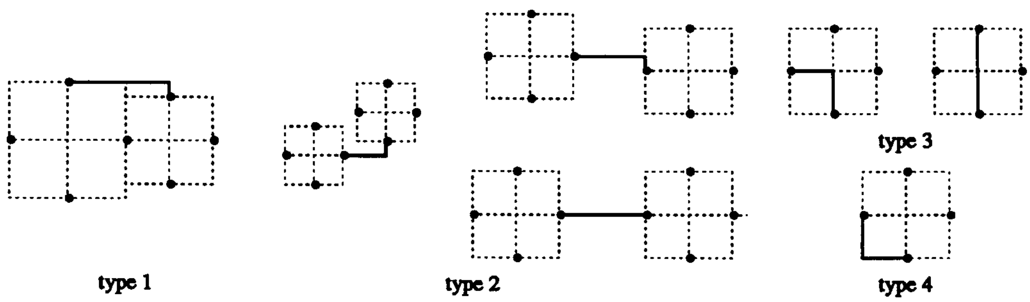


Figure 28: Type of connections

reduce the search space as the produced line segments have one endpoint on the boundary of the smallest enclosing rectangle of the original set V and extend to the adjacent parallel carrier. In this section the concept of clustering is introduced. Considering the original set V, conditions can be given under which it is possible to add line segments to the set of line segments of the eventual approximation tree A. But in contrast to contractions the line segments produced by clustering do not necessarily have an endpoint on the boundary of $\Re(V)$. This implies that the search space is not always reduced.

In this section the clustering is applied to points (which may be virtual) which have a carrier in common.

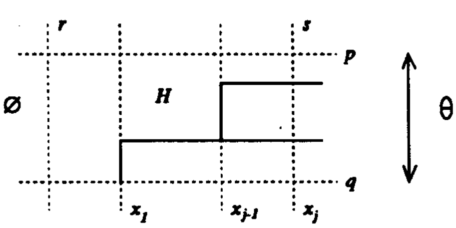
In the first subsection of this section a supporting theorem for the theorems in this section will be proved.

7.1 Reduce Carriers

Definition 15 *A horizontal (vertical)* slab *of width w is a range $(-\infty, \infty) \times (y, y+w)$ for certain y ($(x, x+w) \times (-\infty, \infty)$ for certain x).*

Definition 16 *A horizontal (vertical)* half-slab *of width w is the left (upper) or right (lower) half of a horizontal (vertical) slab of width w, including the side formed by the separating line.*

Let p and q be two horizontal carriers and r and s two parallel carriers perpendicular to p. Let r be the leftmost carrier. Let H be the half-slab having p and q as sides, s as excluded separator, which contains r.



Consider the set L of horizontal line segments of an arbitrary RMST that are located in H. Let v_1, v_2,...,v_j denote the vertical lines through the endpoints of elements of L, numbered from left to right. Obviously $j \leqslant \#L$.

Let $v_i \mathsf{H} v_{i+1}$ ($i \in \{1, ..., j-1\}$) denote the vertical slab between v_i and v_{i+1}, including v_i, but excluding v_{i+1}. Let $v_0 \mathsf{H} v_1$ denote the half plane to the left of v_1, excluding v_1.

Theorem 8 *(Reducing Carriers)*
If $H \cap V = \emptyset$, then every RMST T for V can be rebuilt to form a RMST T' for V which uses at most one carrier parallel to p within H. The parts of T outside H are not affected by this transformation.

Proof By induction.

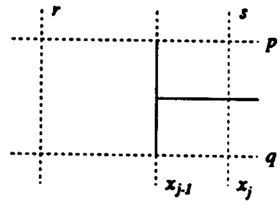


Figure 29: Result of 'Reduce Carriers' in the neighbourhood of j

The theorem will be proved by proving the following: For $i \in \{0, ..., j-2\}$, no horizontal line segment within H is needed in $v_i \mathsf{H} v_{i+1}$ unless it extends to v_{i+2}, but then it is the only horizontal line segment in H. For $i = j-1$, at most one horizontal line segment within H is needed to cover $v_{j-1} \mathsf{H} v_j$.

The statement holds for $i = 0$, because $v_0 \mathsf{H} v_1 \cap L = \emptyset$. No line segment extends to the left of v_1.

Now suppose that the statement holds for $i = 0, ..., k-1$, $k < j-1$. Then $v_0 \mathsf{H} v_k \cap L$ contains no horizontal line segment. Let L_j denote the set of endpoints of elements of L.

- $v_k \cap L_j$ contains no left endpoints. Then the statement holds for $i = 0, ..., k$.
- $v_k \cap L_j$ contains one left endpoint l. Using the facts that $v_0 \mathsf{H} v_k \cap L$ contains no horizontal line segments and T is a RMST, l must possess in v_k an upward connection to p and/or a downward connection to q. The case that l has either an up- or downward connection is solved by a reversion of the L-shape at l.

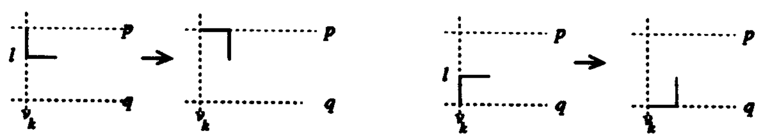
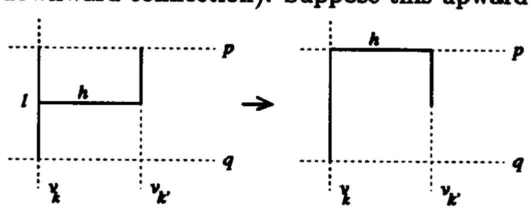


Figure 30: l has either an up- or downward connection

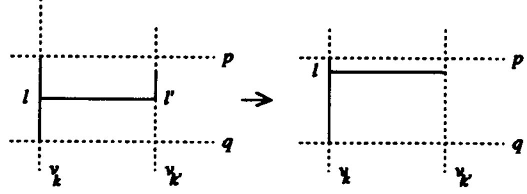
The remaining possibility is that l must possess an upward and a downward connection.

Either ($k = j-1$) the horizontal line segment at l extends through j, or an upward and/or downward connection exists to the horizontal line segment at l, which will be referred to as h. Without loss of generality, assume that this connection is upward. The former possibility satisfies the conditions of the statement. For the latter, suppose there is an upward connection only (analogously for a downward connection). Suppose this upward connection extends to p.

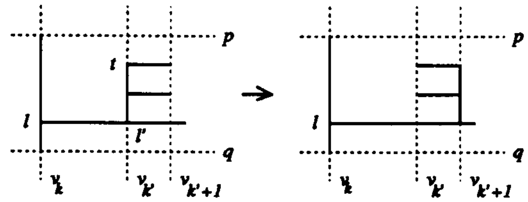


Then shift h upward to p. This proves the statement correct for $i = 0, ..., k$.

Otherwise, if the upward connection $v_{k'}$ does not extend to p, call the lowest point at v_{k+1} b. Let l' be the intersection of h and $v_{k'}$. In this case there is no downward connection so $b = l'$. If there is a L-shape at b, then T is not a RMST, for reversing it reduces $| T |$.

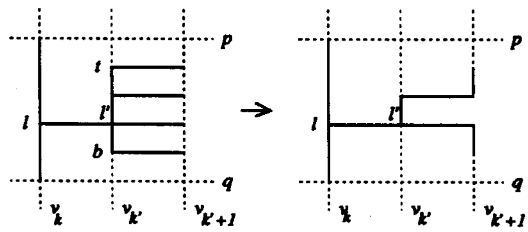


So a downward or right connection must exist as well. Suppose a right connection exists. The upward connection itself may have several connections to the right, but it must have at least one right connection at its top t.



Shifting t•–•l' to the right renders these right connections useless, contradicting the minimality of T. Unless there is a connection to the left and the right connections at t and at l' are the only right connections. But then h can be shifted upward to this left connection, reducing $\mid T \mid$, contradiction.

Suppose a downward connection exists as well. Using considerations analogously to the above, the bottom point b of this vertical line segment $v_{k'}$ must have a right connection. So reversing the L-shapes at t and b renders line segments obsolete.



This contradicts the minimality of T.

- $v_k \cap L_j$ contains two left endpoints. Using the facts that $v_0 H v_k \cap L$ contains no horizontal line segments and T is a RMST, the upper left endpoint t and the lower left endpoint u must connect upward/downward to p/q respectively. If t and u are connected using v_k, then shift t•–•u to the right. Afterwards reverse the two L-shapes. The same reversions are applied when t and u are not connected. These transformations show that the statement holds for $i = 0, ..., k+1$.

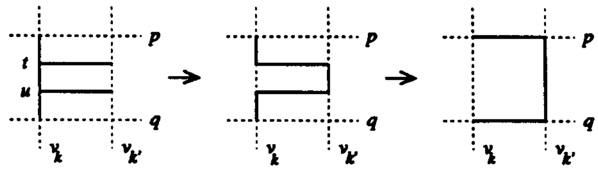


Figure 31: t and u are connected

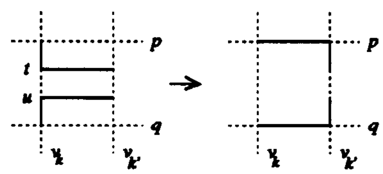
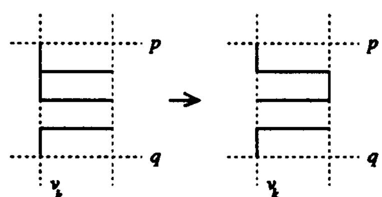


Figure 32: t and u are not connected

- $v_k \cap L_j$ contains three or more left endpoints. Using the same reasoning as above, the upper and lower line segments must be connected to p and q.



Furthermore, because T is a RMST, at least two line segments must be connected via v_k. Shifting these vertical connections to the right produces at least one obsolete line segment, contradicting the minimality of T.

This concludes the proof of the induction step, from which the theorem follows trivially.

□

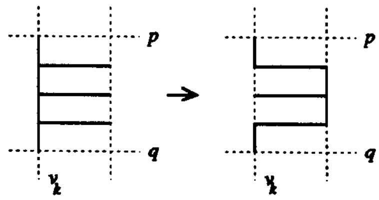


Figure 33: Three left endpoints at v_k

Part of T lying in A is pushed to the boundaries of A by applying 'Reduce Carriers'. The vertical line segments in A will never be shifted to the left. The horizontal line segments are moved up or down.

Corollary 20 *Suppose H is a horizontal half-slab of width Δ on which 'Reduce Carriers' has been applied in RST T' producing RST T. Let $l_1,\ldots,l_m$, for certain m, be the vertical line segments in H. Then T is a RMST implies $\langle l_i, l_{i+1}\rangle \geqslant \Delta$ for all $i \in \{1,\ldots,m-1\}$.*

7.2 Clustering, perpendicular area.

Let p be a carrier with two adjacent vertices, p_i and p_{i+1}, $\langle p_i, p_{i+1}\rangle = \Delta$. Let s_i and s_{i+1} be two carriers perpendicular to p such, that p_i and p_{i+1} are between or on s_i and s_{i+1}. Let A be the slab between s_i and s_{i+1}.

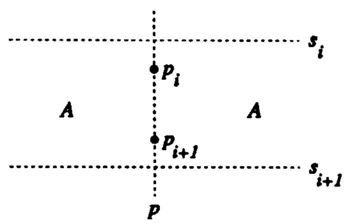
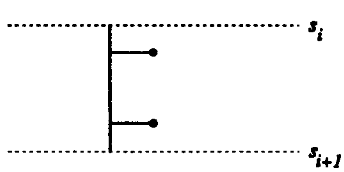


Figure 34: Clustering, perpendicular area

Lemma 11 *If $\langle s_i, s_{i+1}\rangle = \Delta$ and $A \cap V = \varnothing$ then a RMST T for V exists, which contains $p_i \bullet\!\!-\!\!\bullet p_{i+1}$.*

Proof Trivial, for there is no carrier parallel to s_i in A.
□

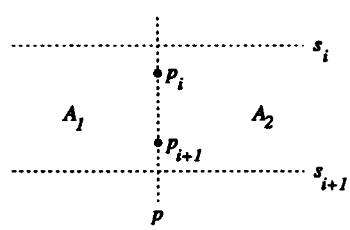
Theorem 9 *If $\langle s_i, s_{i+1}\rangle > \Delta$ and $A \cap V = \{p_i, p_{i+1}\}$ and p_i (p_{i+1}) is not on s_i (s_{i+1}), then every RMST for V must contain $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ or p_i and p_{i+1} are both connected to a line segment l parallel to p in A, such that $| p_i \bullet\!\!\sim\!\!\bullet p_{i+1} | \leqslant 3\Delta$.*



Proof

Suppose p is a vertical carrier. Let A_1 (A_2) denote the half-slab of A to the left (right) of p. Suppose the theorem does not hold. Hence consider a RMST T'' for V, which contradicts the theorem. We will prove T'' is not a RMST.

If in T'' p_i and p_{i+1} are both connected to a line segment l parallel to p in A and $| p_i \bullet\!\!\sim\!\!\bullet p_{i+1} | > 3\Delta$, then T'' is not a RMST, for deleting one of the connections $p_i \bullet\!\!-\!\!\bullet l$, $p_{i+1} \bullet\!\!-\!\!\bullet l$ and adding $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ will produce a RST of shorter total length. Contradiction. Apply 'Reduce Carriers' to T'' with respect to A_1 and A_2, creating a RMST T' for V. T' cannot contain $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ or the theorem would hold.



The path in T' connecting p_i to p_{i+1} cannot contain a vertical line segment without branches in A, or profit would be made by the transformations in claim ??, proclaiming T' is not a RMST. So at least part of a vertical line segment v_1 with horizontal branch h_1 in A_1 or part of a vertical line segment v_2 with horizontal branch h_2 in A_2 is used in T' to connect p_i to p_{i+1}. Three cases remain to be proved:

- no horizontal branch in A is used to connect p_i to p_{i+1}
- one horizontal branch in A is used to connect p_i to p_{i+1}
- both horizontal branches in A are used to connect p_i to p_{i+1}.

• No horizontal branch is used to connect p_i to p_{i+1}. This implies that neither h_1 nor h_2 is attached to p_i or p_{i+1}. As the only two horizontal carriers in A are the ones containing p_i and p_{i+1}, h_1 and h_2 are not on a carrier. Since only RMST's are considered whose line segments are on carriers, this situation will not arise.
• Only h_1 is used to connect p_i to p_{i+1}. (Analogously for h_2).
If h_1 is connected to p_i not using v_1, then shift v_{1_d}, the lower part of v_1, to p, else shift v_{1_u}, the upper part of v_1, to p. Profit will be made, for p_i (if v_{1_u} is shifted) or p_{i+1} (if v_{1_d} is shifted) must be connected to s_i (s_{i+1}) using p.

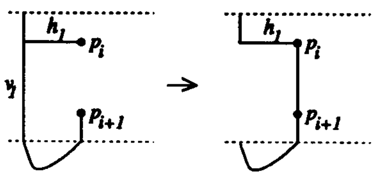


Figure 35: Only h_1 is used to connect p_i to p_{i+1}.

So T' is not a RMST, contradiction.
• Both h_1 and h_2 are used to connect p_i to p_{i+1}. Note that the rightmost point h_{1_r} of h_1 must be p_i or p_{i+1}. The same holds for the leftmost point h_{2_l} of h_2. Otherwise T' would not be a RMST or h_1 (h_2) would not be on a carrier.

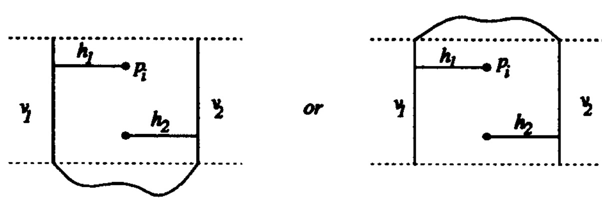


Figure 36: Both h_1 and h_2 are used to connect p_i to p_{i+1}.

Consider $h_{1_r} = p_i$ and $h_{2_l} = p_{i+1}$. (The case $h_{1_r} = p_{i+1}$ and $h_{2_l} = p_i$ can be treated in a similar way.) Adding p_i•–•p_{i+1}creates a cycle containing v_{1_d} or v_{2_u}. So delete v_{1_d} or v_{2_u}, making profit: $| v_{1_d} |> \Delta$ and $| v_{2_u} |> \Delta$.
This implies T' is not a RMST, contradiction.

This concludes the proof of the theorem.

□

Corollary 21 *Suppose there is no carrier parallel to p within distance Δ to p. Then every RMST for V must contain p_i•–•p_{i+1}.*

7.3 Set contraction

Let p_1 be the lowest point of a forest F. Let u_1 (u_5) be the horizontal (vertical) carrier containing p_1. Let p_2 be a point, which may be a virtual point ($p_2 \in I$), lying on u_5, below p_1, such that p_1 and p_2 are adjacent points on u_5. Define $\Delta = \langle p_1, p_2 \rangle$. Let u_2 be the horizontal carrier containing p_2.

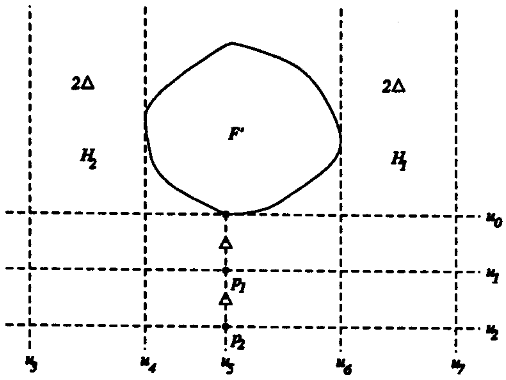
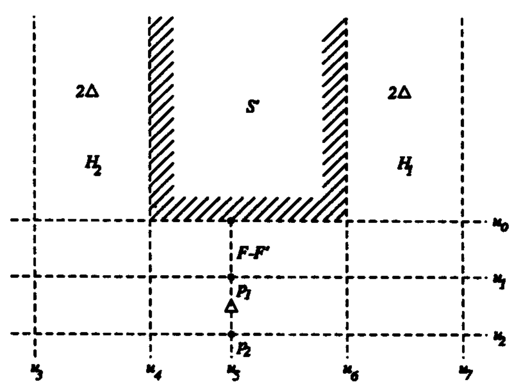


Figure 37: Set contraction

Let F' be a forest, such that every element of F' is a subtree of an element of F and such that its lowest point, on horizontal carrier u_0, has distance at least Δ to u_1. Let $E(F)$ ($E(F')$) denote the collection of edges of elements of F (F'). Let $K(F)$ ($K(F')$) denote the collection of vertices of elements of F (F'). Let F" denote $(E(F) - E(F')) \cup (K(F) - K(F'))$. All elements of F" must lie on u_5, so $W(F") = 0$.
Let u_4 (u_6) be the vertical carrier containing the leftmost (rightmost) point of F. Let u_3 (u_7) be the vertical carrier, which lies 2Δ to the left (right) of u_4 (u_6). Let S be the vertical half-slab of width $4\Delta + W(F)$, containing F, with u_2 as excluded separator, and u_3 and u_7 as excluded sides. Let S' be the vertical half-slab of width $W(F)$, containing F, with u_0 as excluded separator, and u_4 and u_6 as included sides.

Theorem 10 *If $(S - S') \cap V = (F") \cap V$, then a RST T for V exists, which contains $p_1 \bullet\!\!-\!\!\bullet p_2$, and T is minimal among all RST's containing all elements of F.*



Proof Suppose the theorem does not hold. Consider an arbitrary RST T" for V, which is minimal among all RST's containing all elements of F. Let H_1 (H_2) be the vertical half-slab of width 2Δ, having u_3 and u_4 (u_6 and u_7) as sides and u_2 as separator. Apply 'Reduce Carriers' to H_1 and H_2, creating a RST T' for V.

T' cannot contain $p_1 \bullet\!\!-\!\!\bullet p_2$, or the theorem would hold. If T' contains a vertical edge in H_1 (H_2), then call this edge, constrained to H_1 (H_2) v_1 (v_2). The uppermost point of v_1 (v_2) is attached to a horizontal line segment in H_1 (H_2), call this line segment, constrained to H_1 (H_2), h_1 (h_2). So h_1 and h_2 have length 2Δ. The part of h_1 (h_2) to the left of v_1 (v_2) is called lh_1 (lh_2). The following holds: $| rh_i | \geqslant \Delta$ or $| lh_i | \geqslant \Delta$ for $i \in \{1, 2\}$.
T' cannot have an edge e of length Δ or more, such that $e \sqsubset p_1 \bullet\!\!\leadsto\!\!\bullet p_2$. So T' must use part of h_1 or part of h_2 to connect p_1 to p_2, or $p_1 \bullet\!\!\leadsto\!\!\bullet p_2$ does not enter either H_1 or H_2. As rh_i or lh_i has length Δ or more, T' cannot use all of h_1 nor all of h_2. So T' must use v_1 or v_2. Suppose T' uses v_1 (v_2) then T' must use the smallest part of h_1 (h_2).

So four cases remain to be considered:

1. v_1 is used, not v_2, to connect p_1 to p_2
2. v_2 is used, not v_1, to connect p_1 to p_2
3. v_1 and v_2 are used to connect p_1 to p_2
4. $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ does not enter H_1 or H_2

Case 1 v_1 is used, not v_2, to connect p_1 to p_2.
Then $| v_1 | < \Delta$, otherwise the theorem would hold. So v_1 lies below u_1.
• Suppose $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ contains a line segment above u_1. Then lh_1 can extend to the left only so far that the edge containing lh_1 has length less than Δ. This implies that the edge e_1 upward must begin within distance Δ from the topmost point of v_1. The vertical line segment containing e_1 cannot have another horizontal line segment to its right between u_0 and h_1, for then T' is not minimal among all RST's containing all elements of F. If the topmost point of e_1 lies on or above u_0, then e_1 would have length Δ, implying that a tree supporting the theorem does exist, contradiction. So at the top of e_1 there must be a connection e_2 to the left, see the first picture in figure 38. As e_2 reaches to u_5, $e_2 \sqsubset p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$.

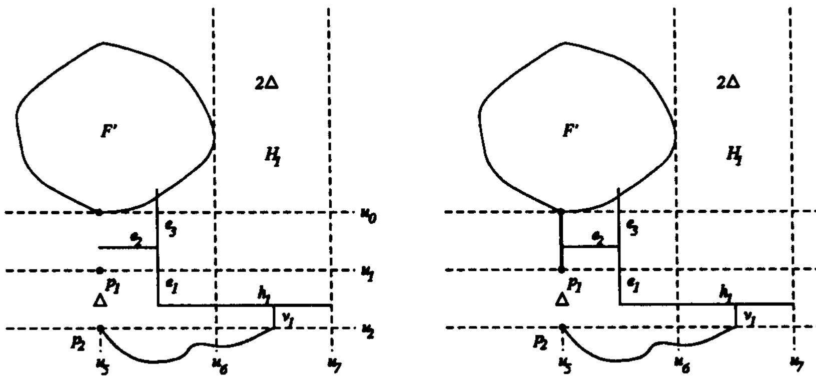


Figure 38: Set contraction, case 1

Furthermore u_0 must be reached by the vertical extension e_3 of e_1, otherwise T' would not be minimal among all RST's containing all elements of F. In T' all elements of F" below e_2 must be in $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$. If the other elements of F" are not connected to e_3 via e_2, then delete e_3 and add a vertical edge from e_2 to the lowest of the elements of F" which lie above e_2. Then reverse the L-shape at the rightmost point of e_2, as p_1 was already connected to the leftmost point of e_2, the reversion of this L-shape reduces the length of T'. Contradiction.
So the elements of F" lying above e_2 must be connected to e_3 via e_2, see the second situation in figure 38. As a result, e_2 can be shifted to u_0 creating a tree in which e_3 and e_1 form one edge of length more than Δ. This implies that deleting e_3 and e_1, then adding $p_1 \bullet\!\!-\!\!\bullet p_2$ creates a tree supporting the theorem, which is of shorter total length than T'. Contradiction.
• No part of $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ lies above u_1.
If $\langle p_1, u_6 \rangle < \Delta$, then there is no connection between the leftmost point of h_1 and F' in $S-(H_1 \cup H_2)$ or T' is not minimal among all RST's containing all elements of F, see figure 39.
This implies that part of $p_1 \bullet\!\!-\!\!\bullet p_2$ already exists with length $\Delta - \mid v_1 \mid$, or this situation can be created by reversing a L-shape. This implies that v_1 can be deleted and $p_1 \bullet\!\!-\!\!\bullet p_2$ minus the existing part can be added, creating a tree supporting the theorem.

So $\langle p_1, u_6 \rangle \geqslant \Delta$, but this implies the existence of an edge e of length Δ or more, such that $e \sqsubset p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$, contradiction.
This concludes the proof of case 1.

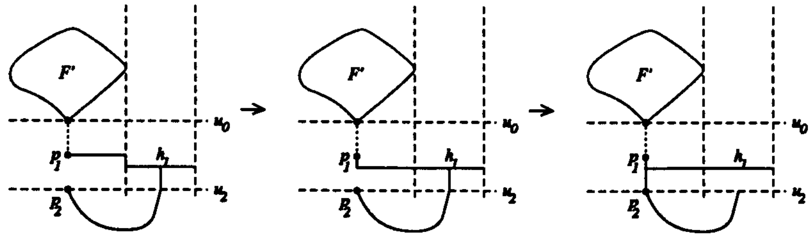
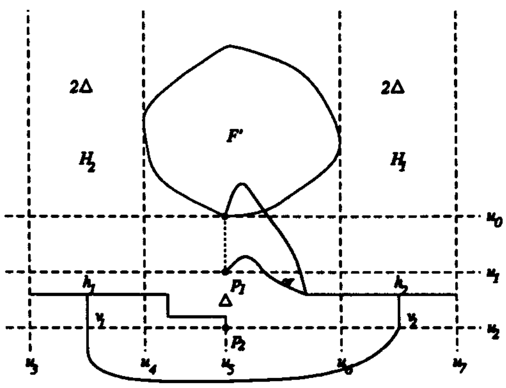
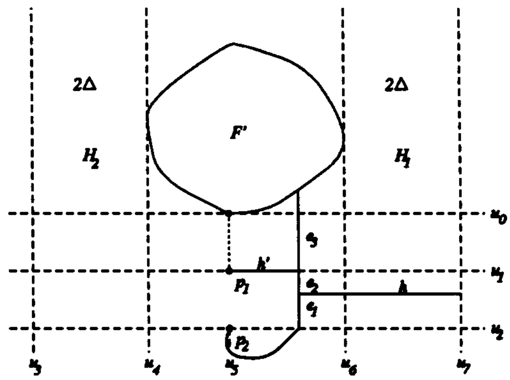


Figure 39: Set contraction: case 1, no part of $p_1 \bullet\!\leadsto\!\bullet p_2$ above u_1.

Case 2 (T' uses v_2 and not v_2 to connect p_1 to p_2.)
This case is similar to case 1.



Case 3 (T' uses v_1 and v_2 to connect p_1 to p_2.)
In this case $| v_1 |$ ($| v_2 |$) $< \Delta$ and there must be a path between the uppermost points of v_1 and v_2 containing neither p_1 nor p_2. Either the leftmost point of h_2 or the rightmost point of h_1 is connected to p_2 without using h_1 or h_2. Suppose p_2 is connected to the rightmost point of h_1, then this case is the same as case 1, otherwise this case is the same as case 2.



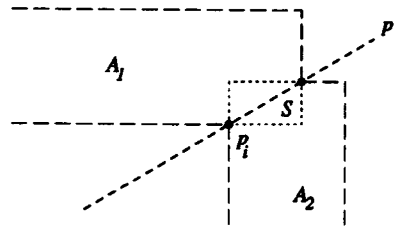
Case 4 ($p_1 \bullet\!\leadsto\!\bullet p_2$ does not enter H_1 or H_2.)
There must be several ($\geqslant 1$) edges in $p_1 \bullet\!\leadsto\!\bullet p_2$ and none of these can have length Δ or more. This implies there must be two edges, e_1 and e_2, in $p_1 \bullet\!\leadsto\!\bullet p_2$ to cover the slab between u_1 and u_2. At their common endpoint a horizontal line segment h must commence, $| h | \geqslant 2\Delta$. This implies there can be no horizontal edge on the same side as h within distance 2Δ above h. So there must be a horizontal edge h' within distance Δ on the other side. Suppose h lies in H_1, then h' lies to the left of e_1 and e_2. There must be a vertical edge e_3 above h', or the L-shape at the rightmost point of h' could be reversed to form part of $p_1 \bullet\!-\!\bullet p_2$ and e_1 would have enough length to complete $p_1 \bullet\!-\!\bullet p_2$. The topmost point of e_3 must be on or above u_0. So h' can be shifted upward to u_0, thus making e_3 part of $p_1 \bullet\!\leadsto\!\bullet p_2$. Analogously to case 1 a tree supporting the theorem can be created. Contradiction. This completes the proof of case 4 and of the theorem.
□

Corollary 22 *If the area to the right of u_6 but above u_2 and the area to the left of u_4 but above u_2 does not contain any elements of V, then the existence of p_2 as an element of V is not needed: there exists a RST T containing the line segment downward from p_1 to u_2, and T is minimal among all RST's containing all elements of F.*

The concept of clustering will be extended to points not necessarily having a carrier in common, this will be called Slanted Clustering.

7.4 Slanted clustering, Γ-shaped area

Let p be any line not parallel to any carrier with two adjacent vertices p_i and p_{i+1}. Let $\Delta = \langle p_i, p_{i+1} \rangle$. Let H_1 and H_2 be half-slabs of width Δ such that p_{i+1} is on both their separating lines and p_i is on one of their other borders, H_1 perpendicular to H_2. Let $S = H_1 \cap H_2$, including all its sides. Let A_1 (A_2) be H_1 (H_2), where all boundaries are excluded.



Theorem 11 *If $(A_1 \cup A_2) \cap V = \varnothing$, then a RMST T for V exists in which the path from p_i to p_{i+1} lies on the boundary of S.*

Proof

Suppose H_1 is a horizontal half-slab and suppose H_2 is a vertical half-slab which uppermost points are on its separating line. Suppose the theorem does not hold. Consider an arbitrary RMST for V, apply 'Reduce Carriers' first to A_1 and then to A_2. Call the created RMST T'. T' cannot have an edge e of length Δ or more, such that $e \sqsubset p_i \bullet\!\!\leadsto\!\!\bullet p_{i+1}$, for then a RMST T can be created which satisfies the theorem.

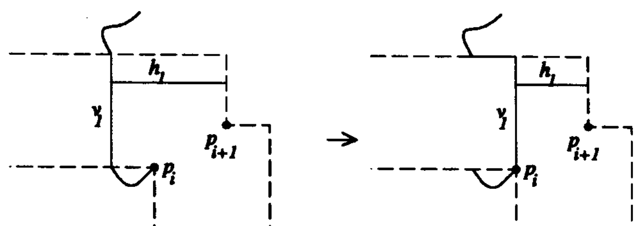
In the following h_i and v_i denote line segments restrained to the region $A_1 \cup A_2$.

Four cases remain to be considered:

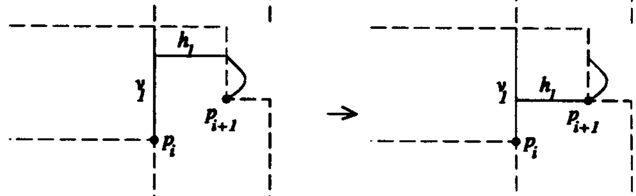
1. Vertical line segment v_1 with branch h_1 in A_1 is used to connect p_i to p_{i+1}.
2. Horizontal line segment h_2 with branch v_2 in A_2 is used to connect p_i to p_{i+1}.
3. Both a part of v_1 and a part of h_2 are used to connect p_i to p_{i+1}.
4. Conflicts as a result of applying 'Reduce Carriers' to perpendicular areas.

Case 1 (Part of v_1 is used in $p_i \bullet\!\!\leadsto\!\!\bullet p_{i+1}$.)

Suppose v_1 lies to the left of p_i. Then move the lower part of v_1 to p_i and reverse the L-shape at the leftmost point h_{1_l} of h_1. So this case is reduced to the case that v_1 is on p_i or v_1 lies to the right of p_i. The latter will be dealt with in case 3 and case 4.



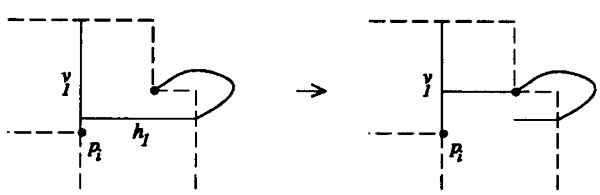
Consider the case that v_1 contains p_i. Suppose $| h_1 | < \Delta$. If h_1 is used to connect p_i to p_{i+1}, then lower h_1 to p_{i+1} to create a tree, which supports the theorem.



If h_1 is not used to connect p_i to p_{i+1}, then add $h_{1_r} \bullet\!\!-\!\!\bullet p_{i+1}$, where h_{1_r} is the rightmost point of h_1. Subsequently h_1 can be lowered to p_{i+1}. Now the upper part of v_1 can be deleted. As this upper part has length $>| h_{1_r} \bullet\!\!-\!\!\bullet p_{i+1} |$, T' cannot be a RMST. Contradiction.

So h_1 must have length Δ. As h_2 is not used to connect p_i to p_{i+1}, $h_1 \neq h_2$.

If h_1 is used to connect p_i to p_{i+1}, then move the part of h_1 which is at the left of p_{i+1} up to p_{i+1}. This way a tree is created in which $p_i \bullet\!\!\leadsto\!\!\bullet p_{i+1}$ lies on the boundary of S. Furthermore, the part of h_1 not being moved is obsolete, implying T' is not a RMST. Contradiction.



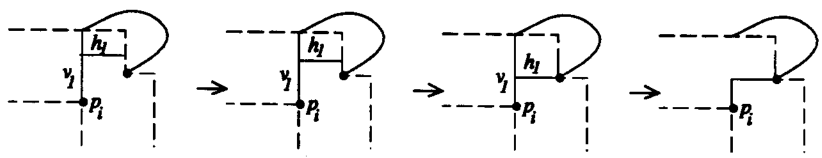
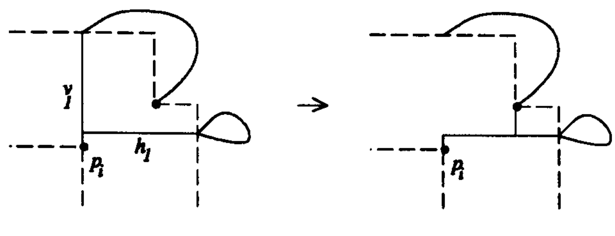


Figure 40: Slanted clustering, Γ-shape: case 1, h_1 is not used and $| h_1 | < \Delta$

If h_1 is not used, then all of v_1 must be in the path from p_i to p_{i+1}. Move the part of v_1 below p_{i+1} but above h_1 to p_{i+1}. The part of v_1 above p_{i+1} is obsolete, so T' is not a RMST. Contradiction. Besides, reversing the L-shape above p_i creates a tree in which p_i is connected to p_{i+1} over the boundary of S.



Case 2 (Part of h_2 is used to connect p_i to p_{i+1}.)
This case is proved analogously to case 1.

Case 3 Both a part of v_1 and a part of h_2 are used to connect p_i to p_{i+1}.
As in case 1, v_1 does not lie to the left of p_i and h_2 does not lie below p_i. In this case $v_1 = v_2$ or $h_1 = h_2$. If $v_1 = v_2$, then shift the part of v_1 between h_1 and h_2 to p_{i+1}.

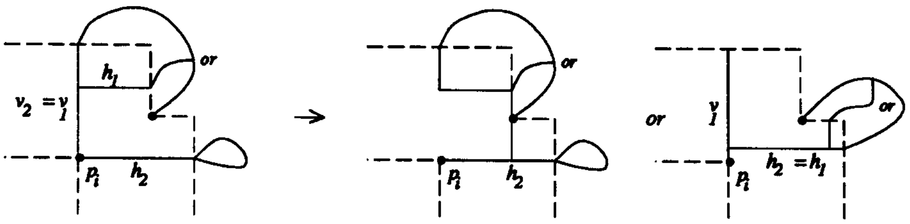


Figure 41: Slanted clustering, Γ-shape: case 3, part of h_2 and v_1 is used and $v_1 = v_2$.

If $h_1 = h_2$, then shift the part of h_1 between v_1 and v_2 to p_{i+1}. This way the path from p_i to p_{i+1} lies on the boundary of S. Furthermore T' is proved to be not a RMST for a cycle is obtained. Contradiction.

Case 4 (Conflicts as a result of applying 'Reduce Carriers' to perpendicular areas.)
After applying 'Reduce Carriers' to A_1 there could be no horizontal line segment in A_1. Suppose that after applying 'Reduce Carriers' to A_2 there is a vertical line segment v_2 in A_2, then v_2 cannot be to the left of p_{i+1}. If v_2 contains p_{i+1}, then reversing the L-shape at the leftmost point of h_2 renders a tree satisfying the theorem. The only situation not treated above is that h_2 is above p_i. This implies that a L-shape has been reversed so that its corner lies in A_1. If v_2 is in the connection from p_i to p_{i+1}, then shift v_2 to p_{i+1} and reverse the mentioned L-shape again, creating a tree T supporting the theorem.
Otherwise add p_{i+1}•–•v_{2_u}, where v_{2_u} is the uppermost point of v_2, then shift v_2 to p_{i+1}. Subsequently delete the part of h_2 to the right of v_2; this part has greater length than v_{2_u}•–•p_{i+1}. This implies T' is not a RMST. Contradiction.
If there is a horizontal line segment in A_1 after applying 'Reduce Carriers' to A_1, then $h1$ could be above, below or on p_{i+1}. Suppose h_1 contains p_{i+1} then T' supports the theorem, contradiction. So h_1 is below or above p_{i+1}. First suppose h_1 is above p_{i+1}, then this configuration can only be affected by the application of 'Reduce Carriers' to A_2 if v_1 is to the right of p_i. This implies that

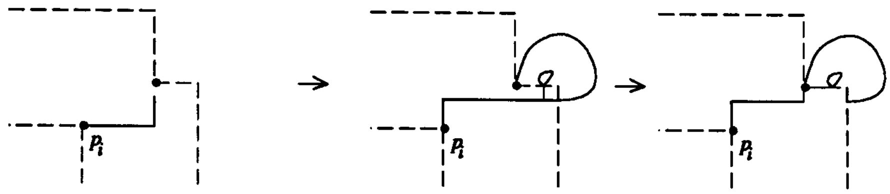


Figure 42: Slanted clustering, Γ-shape: case 4, h_2 is above p_i

this application will reverse the L-shape at the bottom point of v_1 or a configuration is created as treated in the previous cases.

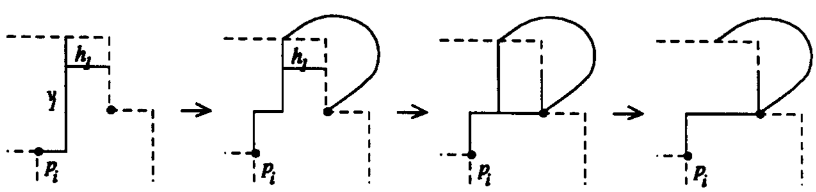


Figure 43: Slanted clustering: case 4, h_1 is above p_{i+1}

If $h_1 \sqsubset p_i \bullet\!\!\rightsquigarrow\!\!\bullet p_{i+1}$, then lowering h_1 to p_{i+1} will create a tree T supporting the theorem. Contradiction. Otherwise add $h_{1_r} \bullet\!\!-\!\!\bullet p_{i+1}$, where h_{1_r} is the rightmost point of h_1, then lower h_1. Now the vertical line segment v' above p_{i+1}, which is in the created cycle, can be deleted. As $| v' | > | h_{1_r} \bullet\!\!-\!\!\bullet p_{i+1} |$, T' cannot be a RMST. Contradiction.

So suppose h_1 is below p_{i+1}. The same reasoning as above holds, so v_2 does not exist. Call the upper part of the original v_1 v'.

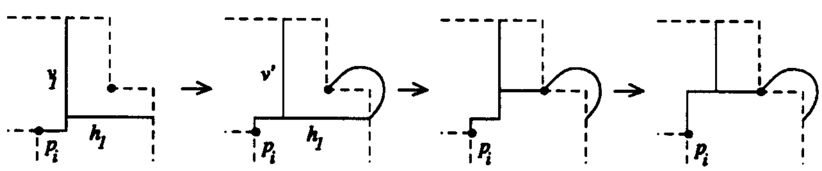


Figure 44: Slanted clustering: case 4, h_1 is below p_{i+1} and h_1 is used

If h_1 is used to connect p_i to p_{i+1}, then delete h_1 to add a horizontal line segment h' from p_{i+1} to v'. As $| h_1 | > | h' |$, T' is not a RMST. Contradiction.

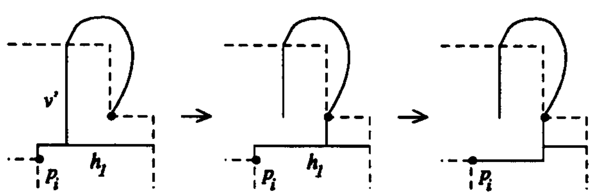


Figure 45: Slanted clustering: case 4, h_1 is below p_{i+1} and h_1 is not used

So v' must be in the connection from p_i to p_{i+1}. Shift the part of v' below p_{i+1} but above h_1 to p_{i+1}. Reverse the L-shape at the leftmost point of h_1. This way a tree is created which supports the theorem. Furthermore the part of v' that has not been shifted is obsolete. This implies that T' is not a RMST, contradiction.

This concludes the proof of the theorem. □

If the points considered do have a carrier in common a stronger lemma immediately follows from theorem 11. For the boundary of S is $p_i \bullet\!\!-\!\!\bullet p_{i+1}$.

Corollary 23 *If p_i and p_{i+1} have a carrier in common and $(A_1 \cup A_2) \cap V = \emptyset$ then a RMST T for V exists, which contains $p_i \bullet\!\!-\!\!\bullet p_{i+1}$.*

7.5 Slanted clustering, #-shaped area

Let p_1, p_2 be elements of V, such that p_1 lies to the left and above p_2 (or rotations over $90°$). Let u_1 (u_2) be the horizontal carrier, which contains p_1 (p_2). Let q_1 (q_2) be the vertical carrier, which contains p_1 (p_2). Let $b = \langle q_1, q_2 \rangle$ and $h = \langle u_1, u_2 \rangle$. Let u_0 (u_3) be a carrier parallel to u_1 which lies b above (below) u_1 (u_2). Let u_4 (u_5) be a carrier parallel to q_1, which lies $b + h$ (h) to the left of q_1. Let u_6 (u_7) be a carrier parallel to q_2, which lies h ($b + h$) to the right of q_2, see figure 46.
Define D to be the vertical half-slab of width b with q_1 and q_2 as included sides and u_3 as separator, such that D does not contain p_2.
Define B_1 (B_2) to be the vertical slab of width b or more, which has u_4 and u_5 (u_6 and u_7) as excluded sides. B_1 (analogously for B_2) is needed, because then the left part of horizontal line segments in the slab between u_4 and q_1 always has length at least b, enough to cross the slab between q_1 and q_2. This way the distance between u_4 and q_1 can be reduced to $b + h$, as compared to $2b + h$, the distance between u_4 and u_5 in $\cap$-shaped slanted clustering.
Define B_3 (B_4) to be the horizontal slab of width b or more, which has u_0 and u_1 (u_2 and u_3) as excluded sides.
Define B to be the union of all B_i, $(1 \leqslant i \leqslant 4)$.
Define A to be the vertical half-slab of width $3b + 2h$, which has u_4 and u_7 as excluded sides and u_0 as excluded separator and which contains p_1.

Theorem 12 *If $((A - D) \cup B) \cap V = \{p_1, p_2\}$ then there exists a RMST T for V in which $| p_1 \bullet\!\!\sim\!\!\bullet p_2 | = b + h$.*

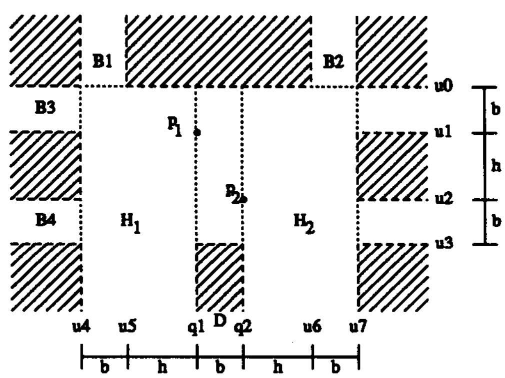


Figure 46: #-shaped area

Proof
Notice that all slabs B_i contain only line segments (of length b) perpendicular to the sides of the slabs. Suppose the theorem does not hold. Consider a RMST T' for V in which $| p_1 \bullet\!\!\sim\!\!\bullet p_2 | > b + h$.
Define H_1 (H_2) to be the vertical half-slab of width $b + h$ having u_4 and q_1 (q_2 and u_7) as sides and u_0 as separator, which lies in A.
Apply 'Reduce Carriers' to H_1 and H_2. If T' uses a vertical line segment v_1 (v_2) in H_1 (H_2), then $\langle v_1, q_1 \rangle \leqslant h$ ($\langle v_2, q_2 \rangle \leqslant h$). Let v_1 (v_2) be attached to h_1 (h_2) a horizontal line segment in H_1 (H_2) constrained to H_1 (H_2).

Suppose for T' still holds $| p_1 \leadsto p_2 | > b + h$. Then T' cannot have an edge e of length $b + h$ or more, such that $e \sqsubset p_1 \leadsto p_2$. Otherwise e can be deleted and a path between p_1 and p_2 of length $b + h$ can be created, contradiction.
If v_1 (v_2) exists, then $| v_1 | \geqslant b$ ($| v_2 | \geqslant b$), for slab B_3 has width b or more.
Four cases remain to be considered:

1. no part of either h_1 or h_2 is used to connect p_1 to p_2
2. no part of h_1 is used to connect p_1 to p_2, but part of h_2 is.
3. no part of h_2 is used to connect p_1 to p_2, but part of h_1 is.
4. both part of h_1 as well as part of h_2 is used to connect p_1 to p_2

Case 1 (No part of either h_1 or h_2 is used to connect p_1 to p_2)
T' did not have a $| p_1 \leadsto p_2 | = b + h$, therefore p_1 must be connected to p_2 via D or the area above u_1.

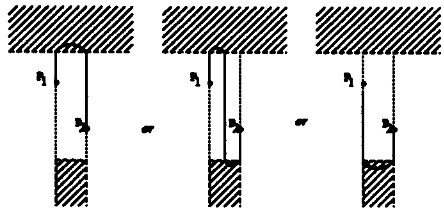


Figure 47: Slanted clustering, #-shape: case 1.

In each case there is an edge of length at least b in the path from p_1 to p_2. This is enough to create a path $p_1 \leadsto p_2$ of length $b + h$. This contradicts the assumption that no tree with this property exists.
Case 2 (No part of h_1 is used to connect p_1 to p_2)

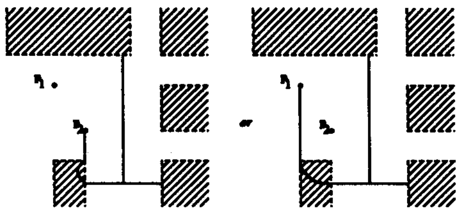


Figure 48: Slanted clustering, #-shape: case 2.

- h_2 is below or on u_2
- h_2 is above u_2

• If h_2 is below or on u_2 then v_2 is not used in the path $p_1 \leadsto p_2$ for v_2 has length at least $b + h$. Suppose p_2 is connected to the leftmost point of h_2 without use of h_2, as depicted in figure 49. The following transformation will build a tree supporting the theorem. Move the left part of h_2 to u_0. Move v_2 to q_2 and use the right part of h_2 which has length at least b to connect p_1 to the image of p_1 on q_2.
So p_2 must be connected to the leftmost point of h_2 using h_2. This implies that p_1 is connected to D using a vertical line segment and p_2 is connected to the area above u_0 using a vertical edge e of length at least $b + h$, as depicted in figure 50. Delete edge e and add a horizontal edge between p_2 and the vertical line segment downward from p_1 to create a tree of shorter total length. This contradicts the fact that T' is RMST, contradiction.

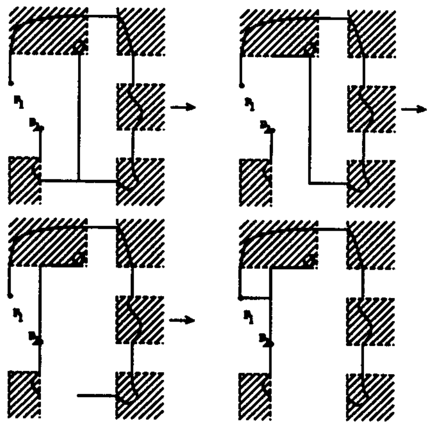


Figure 49: Slanted clustering, #-shape: case 2, p_2 is directly connected to the leftmost point of h_2.

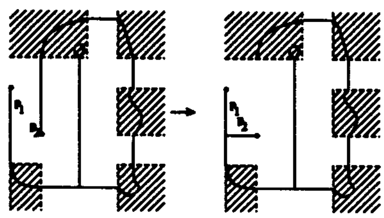


Figure 50: Slanted clustering, #-shape: case 2, p_1 is directly connected to the leftmost point of h_2.

• So h_2 must be above u_2 and the possibility that v_2 is used in the path $p_1 \leadsto p_2$, must be considered.

First suppose v_2 is not used to connect p_1 to p_2. And suppose p_1 is connected to the left-most point of h_2 via h_2, see figure 51. Then p_2 is connected to the leftmost point of h_2 using a vertical edge upward from p_2, otherwise T' would not be a RMST. This implies that the same transformation as depicted in figure 49 creates a tree supporting the theorem. Contradiction.

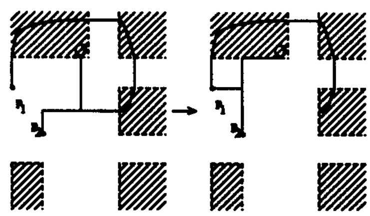


Figure 51: Slanted clustering, #-shape: case 2, h_2 above u_2, p_1 is directly connected to the leftmost point of h_2.

So if v_2 is not used to connect p_1 to p_2, then p_1 must be connected to the leftmost point of h_2 without use of h_2. But then p_2 must be connected to the rightmost point of h_2 using a horizontal edge (of length at least $b+h$) through H_2 or using part of h_1. The latter will be dealt with in case 4, the first has already been discussed.

So v_2 must be used to connect p_1 to p_2. For the same reasons as above, p_1 must be connected to the leftmost point of h_2 using h_2. So p_2 is connected to the leftmost point of h_2 using a vertical edge e upward from p_2, see figure 52.

Move the part of v_2 below u_1 to q_2, then use the rest of v_2 (which has length b) to connect p_1 to the image of p_1 on q_2. This way a tree is created which supports the theorem, contradiction.

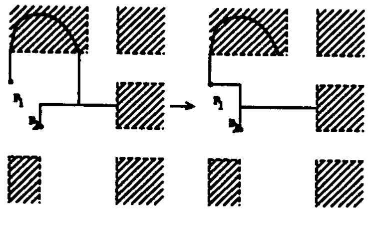


Figure 52: Slanted clustering, #-shape: case 2, h_2 above u_2, v_2 is used.

This concludes the proof of case 2.

case 3 (No part of h_2 is used to connect p_1 to p_2.)

- h_2 is below or on u_2
- h_2 is above u_2

• suppose h_1 lies below or on u_2. This implies that v_1 is not part of the path between p_1 and p_2, for v_1 has length $b + h$ or more. In this case it is immaterial whether p_1 or p_2 is connected to the rightmost point of h_1 using h_1. The following transformations will create a tree supporting the lemma: move the right part of h_1 to u_0, in order to connect to uppermost point of v_1 to q_1. Then move the part of v_1 between u_0 and u_1 to q_1, thus connecting the subtree attached to the uppermost point of the original v_1 to p_1. Now use the left part of h_1 and the rest of v_1, which together have length at least $b + h$, to create a path between p_1 and p_2 of length $b + h$. These transformations have been depicted in figure 53. Close inspection will show that T' cannot be a RMST, but as this fact is not needed to prove the theorem it was not elaborated upon.

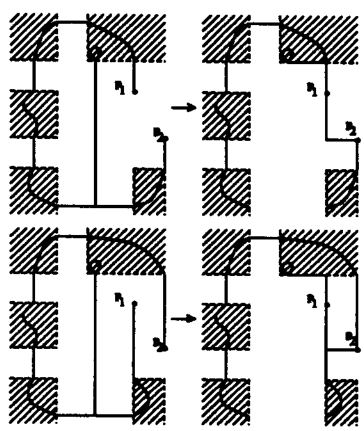


Figure 53: Slanted clustering, #-shape: case 3, h_1 is below or on u_2.

• So suppose h_1 to be above u_2. Then p_1 is connected to the rightmost point of h_1 using h_1 or p_1 is the rightmost point of h_1, otherwise T' would not be a RMST.
If p_1 is the rightmost point of h_1, then p_2 must be connected to the area above u_0 using a vertical line segment upward from p_2. This vertical edge contains a part of length b, namely the part that crosses B_3. This part can be used to connect p_1 horizontally to the rest of mentioned vertical line segment. This way a tree is created which supports the theorem, contradiction.

So p_1 is not the rightmost point of h_1 and p_1 is connected to the rightmost point of h_1 using h_1, see figure 54. Now suppose v_1 is used to connect p_1 to p_2. Then moving the part of v_2 below u_1 to q_1 connects p_1 via this part to the rightmost point of h_1. If this creates the required tree then the rest of v_1 is obsolete, so T' is not a RMST. Otherwise the rightmost point of h_1 must be connected to p_2 using a vertical edge downward from h_1 to D. Part of this edge crosses B_4, enough to create a horizontal connection from the rest of this edge to p_2. This renders a tree supporting the theorem and leaves the upper part of v_1 obsolete. So T' is not a RMST, contradiction.

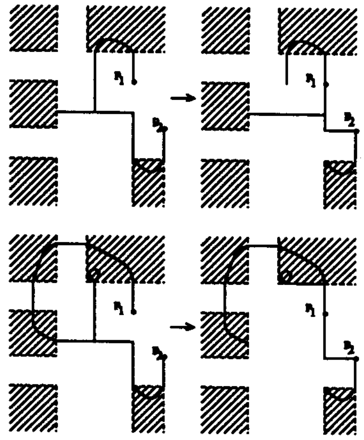


Figure 54: Slanted clustering, #-shape: case 3, h_1 above u_2.

The remaining case is that v_1 is not used to connect p_1 to p_2. Then move the right part of h_1 to u_0, thus connecting p_1 to the uppermost point of v_1. Then move v_1 to q_1, this creates vertical connection between p_1 and the rightmost point of h_1. The left part of h_1 has become obsolete, so T' is not a RMST. Contradiction. Furthermore a tree supporting the theorem can be created in the same way as in the case v_1 was used to connect p_1 to p_2.
This concludes the proof of case 3.

case 4 (Both a part of h_1 and of h_2 is used to connect p_1 to p_2.)
There are four possibilities to be considered:

- h_1 and h_2 below or on u_2
- h_1 below or on u_2 and h_2 above u_2
- h_1 above u_2 and h_2 below or on u_2
- h_1 and h_2 above u_2

• h_1 and h_2 below or on u_2.
In this case neither v_1 nor v_2 is used to connect p_1 to p_2, for both these edges have length at least $b+h$. The only cases that can arise in a RMST are that the leftmost point lh_2 of h_2 is connected to the rightmost point rh_1 of h_1 without using either h_1 or h_2 or that the leftmost point lh_1 of h_1 is connected to the rightmost point rh_2 of h_2 without using either h_1 or h_2, see figure 55.

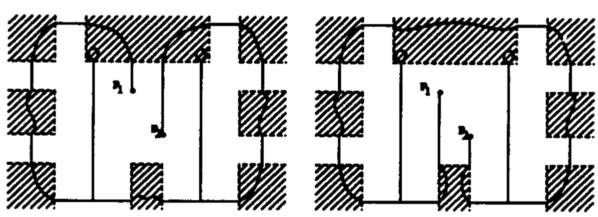


Figure 55: Slanted clustering, #-shape: case 4, h_1 and h_2 below or on u_2.

In the first case p_1 is connected to p_2 using a vertical edge upward from p_1 and a vertical edge upward from p_2. In the latter case these vertical edge go downward from p_1 and p_2. In the first (latter) case one of these vertical edges crosses B_3 (B_4) and has length b, delete this edge to connect p_1 (p_2) to the other vertical edge, creating a path between p_1 and p_2 of length $b + h$.

- h_1 below or on u_2 and h_2 above u_2.

In this case p_1 must be connected to the leftmost point of h_2 using h_2 unless h_2 lies on u_1, for otherwise T' would not be a RMST, see figure 56.

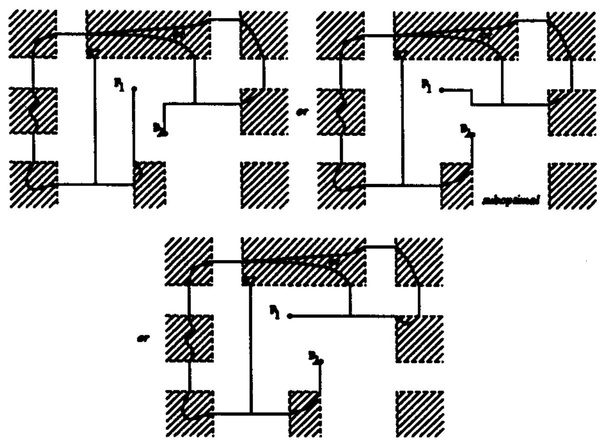
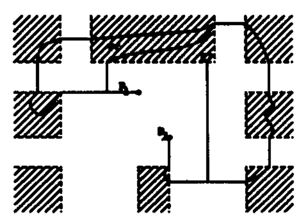


Figure 56: Slanted clustering, #-shape: case 4, h_1 below or on u_2 and h_2 above u_2.

However the transformations given in case 3 are applicable. Contradiction.

- h_1 above u_2 and h_2 below or on u_2.



In this case p_1 must be connected to the leftmost point of h_2 using h_2, otherwise T' would contain a cycle, contradiction. This implies that h_1 must lie on u_1 instead of below u_1, otherwise T' would not be a RMST, see the adjacent picture. However the transformations given in case 2 are applicable.

- h_1 and h_2 above u_2

For the same reasons as in the subcase above, p_1 must be connected to the leftmost point of h_2 using h_2 and h_1 must be on u_1. This implies that the transformations used in case 2 are applicable. Contradiction.

This completes the proof of case 4 and the proof of the theorem.

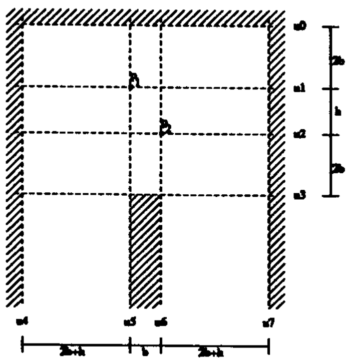
□

The following lemma is an instance of theorem 12.

Lemma 12 *If p_1 and p_2 have a carrier in common and $(A - D) \cap V = \{p_2\}$, then a RMST T for V containing $p_i \bullet\!\!-\!\!\bullet p_{i+1}$ exists.*

Additional slanted clusterings, in which some of the slabs B_1, B_2, B_3 and B_4 can be omitted in exchange for other areas, can be proved in a similar way.

Let p_1 be a vertex on horizontal carrier u_1 and on vertical carrier u_5. Let p_2 be a vertex on horizontal carrier u_2 and on vertical carrier u_6. Without loss of generality assume u_1 to be above u_2 and u_5 to the left of u_6. Let $h = \langle u_1, u_2 \rangle$ and let $b = \langle u_5, u_6 \rangle$. Let S denote $\Re(\{(p_{1_x}, p_{1_y}+b), (p_{2_x}, p_{2_y}-b)\})$, upper and lower boundary excluded, the other boundaries included. Define u_3 (u_0) as the horizontal carrier $2b$ below (above) u_2 (u_1). Let u_4 (u_7) be the vertical carrier $2b+h$ to the left (right) of u_5 (u_6).



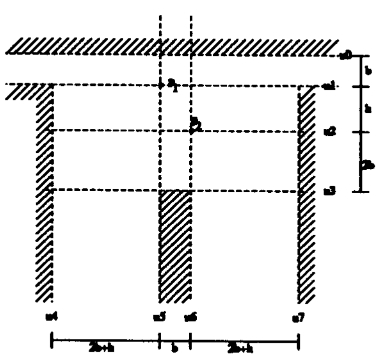
Let H denote the vertical half-slab of width $5b+2h$ having u_4 and u_7 as sides and u_0 as separator, which contains p_1 and in which its separator is excluded. Let H' denote the vertical half-slab of width b, having u_5 and u_6 as sides and u_3 as separator, which does not contain p_1 and in which all sides are included.

Lemma 13 *If* $(H - H') \cap V = \{p_1, p_2\}$ *then a RMST T for V exists, in which* $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ *lies in S. So in T* $\mid p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2 \mid < 3b + h$.

Proof The proof of this lemma is similar to the proof of theorem 12. □

Corollary 24 *If* $u_1 = u_2$ *and* $(H - H') \cap V = \{p_1, p_2\}$, *then a RMST T for V exists, such that T contains* $p_1 \bullet\!\!-\!\!\bullet p_2$.

Let p_1 be a vertex on horizontal carrier u_1 and on vertical carrier u_5. Let p_2 be a vertex on horizontal carrier u_2 and on vertical carrier u_6. Without loss of generality assume u_1 be above u_2 and u_5 to the left of u_6. Let $h = \langle u_1, u_2 \rangle$ and let $b = \langle u_5, u_6 \rangle$. Let S denote $\Re(\{p_1, (p_{2_x}, p_{2_y} - b)\})$, all boundaries but the lower one included. Define u_3 (u_0) as the horizontal carrier $2b$ (b) below (above) u_2 (u_1). Let B be the slab with u_0 and u_1 as sides. Let u_4 (u_7) be the vertical carrier $2b+h$ to the left (right) of u_5 (u_6).



Let H denote the vertical half-slab of width $5b+2h$ having u_4 and u_7 as sides and u_0 as separator, which contains p_1 and in which its separator is excluded. Let H' denote the vertical half-slab of width b, having u_5 and u_6 as sides and u_3 as separator, which does not contain p_1 and in which all sides are included.

Lemma 14 *If* $((H - H') \cup B) \cap V = \{p_1, p_2\}$ *then a RMST T for V exists, in which* $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ *lies in S. So in T* $\mid p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2 \mid \leqslant 3b + h$.

Corollary 25 *If* $u_1 = u_2$ *and* $((H - H') \cup B) \cap V = \{p_1, p_2\}$, *then a RMST T for V exists, such that T contains* $p_1 \bullet\!\!-\!\!\bullet p_2$.

7.6 Slanted clustering, Z-shaped area

Let p_1 (p_2) be a vertex on horizontal carrier u_0 (u_1) and on vertical carrier u_4 (u_3). Rotations over 90° are allowed. Suppose p_1 lies above and to the right of p_2. Define $b = \langle u_3, u_4 \rangle$ and $h = \langle u_0, u_1 \rangle$. Define u_2 (u_5) to be the vertical carrier which lies $b+h$ to the left (right) of u_4 (u_3). Let H_1 (H_2) be the vertical half-slab of width $b+h$ which has u_3 and u_5 (u_2 and u_4) as excluded sides and u_0 (u_1) as excluded separator, such that p_2 (p_1) lies on its left (right) side. This situation is depicted in figure 57.

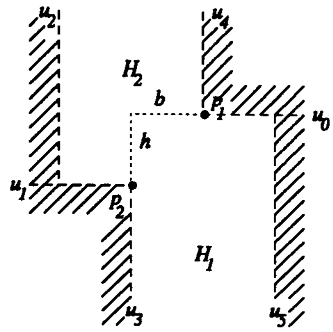


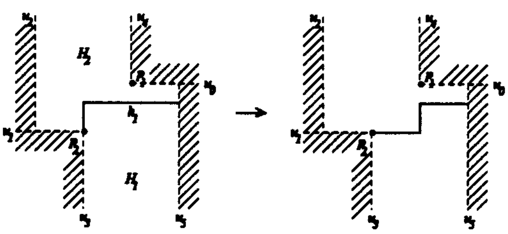
Figure 57: Slanted clustering, Z-shape.

Theorem 13 *If* $(H_1 \cup H_2) \cap V = \varnothing$ *then there exists a RMST T for V in which* $| \; p_1 \rightsquigarrow p_2 \; | = \langle p_1, p_2 \rangle$.

Proof

Suppose the theorem does not hold. Consider an arbitrary RMST T' for V. In T' apply 'Reduce Carriers' to H_1 and H_2.

If 'Reduce Carriers' is applied first to H_1 and then to H_2, a situation can arise in which, after the application of 'Reduce Carriers' to H_1, a L-shape enters H_2. The application of 'Reduce Carriers' to H_2 will reverse this L-shape so that it will lie in H_1. In the rest of this proof, the reversed L-shape will be considered to be reversed again, whenever this is convenient.



After the application of 'Reduce Carriers', T' cannot contain a path $p_1 \rightsquigarrow p_2$ of length $b + h$ or the theorem would hold.

If a vertical edge v_1 (v_2) is used in H_1 (H_2), then v_1 (v_2) is attached to a horizontal line segment h_1 (h_2), constrained to H_1 (H_2). The part of h_1 (h_2) to the left of v_1 (v_2) will be referred to as lh_1 (lh_2). Analogously rh_1 (rh_2) is the part of h_1 (h_2) which lies to the right of v_1 (v_2). Note that neither v_1 nor v_2 can lie between u_3 and u_4.

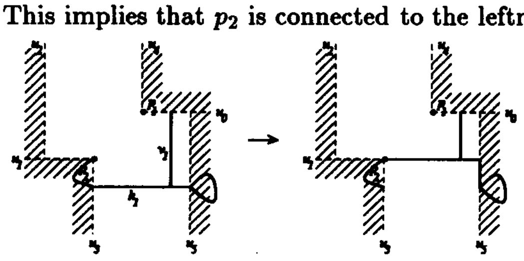
T' cannot contain an edge e of length $b + h$, such that $e \sqsubset p_1 \rightsquigarrow p_2$, otherwise delete e and add a connection between p_1 and p_2 of length $b + h$, thus creating a tree supporting the theorem. This implies that at least part of h_1 or part of h_2 must be used in T' to connect p_1 to p_2.

Three cases remain to be considered:

1. no part of h_2 is used to connect p_1 to p_2
2. no part of h_1 is used to connect p_1 to p_2
3. Both a part of h_1 and a part of h_2 is used to connect p_1 to p_2

Case 1 (no part of h_2 is used to connect p_1 to p_2)

This implies that p_2 is connected to the leftmost point of h_1 without using h_1.



If h_1 lies below u_1, then moving lh_1 up to u_1 and reversing the created L-shape at the lowest point of v_1, will create a situation in which h_1 lies on u_1.

So h_1 must lie on or above u_1. There are two subcases to be considered:

- $v_1 \sqsubset p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$
- $v_1 \not\sqsubset p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$

- First suppose $v_1 \sqsubset p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ as depicted in figure 58.

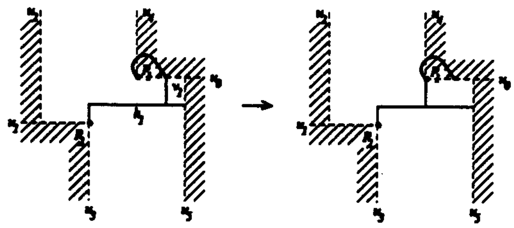


Figure 58: Slanted clustering, Z-shape: case 1, v_1 used

Then move v_2 to u_4 to create a tree supporting the theorem. Contradiction.

- So v_1 cannot be part of $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$.

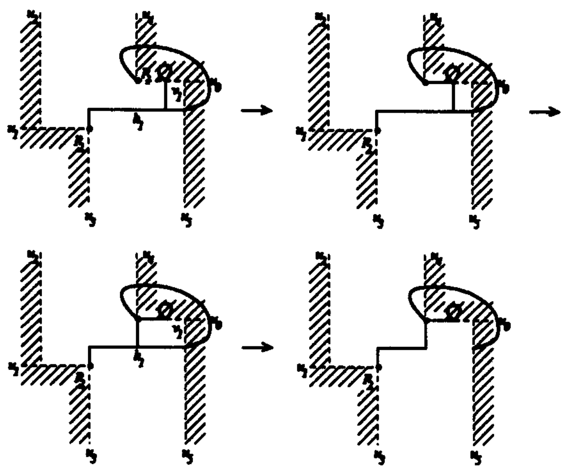


Figure 59: Slanted clustering, Z-shape: case 1, v_1 not used

Then the following transformations, depicted in figure 59, will create a tree supporting the theorem. Add $p_1 \bullet\!\!-\!\!\bullet v_{1_t}$, where v_{1_t} denotes the uppermost point of v_1. This creates a cycle, such that v_1 is part of a connection between p_1 and p_2. Shift v_1 to u_4, now a path $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ of length $b + h$ is created. Delete the part of h_1 which lies to the right of u_4. The length of this part is more than enough to cover the costs of adding $p_1 \bullet\!\!-\!\!\bullet v_{1_t}$. This implies that T' is not a RMST, additional contradiction.
This concludes the proof of case 1.

Case 2 (no part of h_1 is used to connect p_1 to p_2)
When rotated over 180° this case is exactly the same as case 1.

Case 3 (both a part of h_1 and of h_2 are use to connect p_1 to p_2.)
The following subcases have to be considered:

- h_1 lies below u_1 and h_2 lies above u_0
- h_1 lies above u_1 and h_2 lies above u_0
- h_1 lies below u_1 and h_2 lies below u_0

- h_1 lies above u_1 and h_2 lies below u_0

- h_1 below u_1 and h_2 above u_0.

This implies that at least v_1 or v_2 is in the path from p_1 to p_2, otherwise H_1 or H_2 must be crossed again, so T' would contain an edge of length $b+h$ in $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$. Suppose lh_1 is not part of $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$. The following transformations, depicted in figure 60, create a tree supporting the theorem.

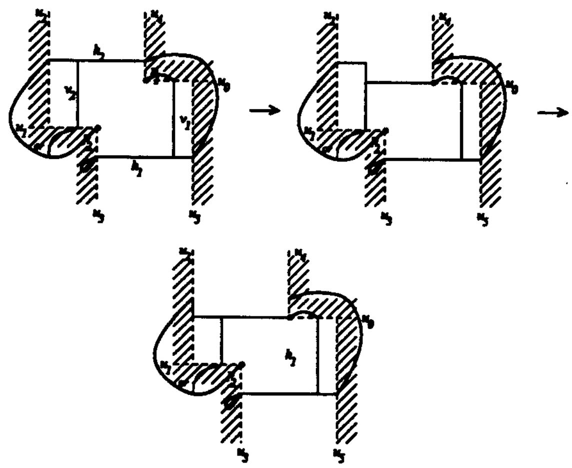


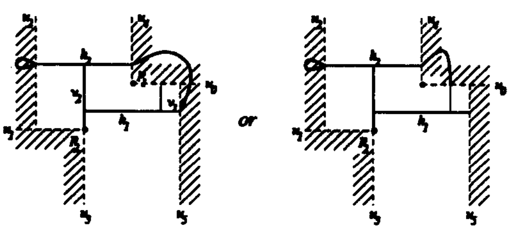
Figure 60: Slanted clustering, Z-shape: case 3, h_1 below u_1 and h_2 above u_0.

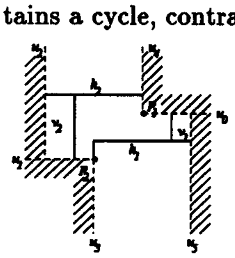
Move rh_2 down to u_0, this way a tree is created in which no part of h_1 is used to connect p_1 to p_2. If the L-shape at the topmost point of v_2 is reversed, the created tree satisfies case 2. So a tree supporting the theorem can be created, contradiction.

In figure 60 lh_1 is not part of $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ in T', but rotating the figure over 180° renders the only other possibility to use h_1 and h_2 in $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$ with h_1 below u_1 and h_2 above u_0.

- Consider the situation that h_1 lies above u_1 and h_2 lies above u_0.

Suppose v_2 lies on u_3, as in the adjacent picture, then the leftmost point of h_1 lies on v_2. In this situation lh_2 cannot be part of $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$, for then H_1 or H_2 should be crossed again, implying there is an edge of length $b+h$ in $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$. If h_1 and h_2 are to be used in this path, then there must be a connection between the rightmost point of h_2 and either v_1 or the rightmost point of h_1. In both cases T' contains a cycle, contradiction.



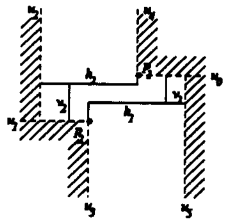


So v_2 must lie to the left of u_3. As the leftmost point of h_1 is connected to p_2 without using h_1, rh_1 or v_1 must be connected to the rightmost point of h_2, otherwise H_1 or H_2 must be crossed again. But then H_1 or H_2 must be crossed again as well to reach p_1. So if both a part of h_1 and a part of h_2 are used in $p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$, with h_1 above u_1 and h_2 above u_0, then an edge e of length $b+h$ in T' exists, such that $e \sqsubset p_1 \bullet\!\!\rightsquigarrow\!\!\bullet p_2$. Contradiction.

- h_1 lies below u_1 and h_2 lies below u_0.

When rotated over 180°, this subcase is exactly the same as the previous subcase.

- So h_1 must lie above u_1 and h_2 must lie below u_0.



Suppose h_1 and h_2 lie on different horizontal carriers. To ensure that both h_1 and h_2 are used to connect p_1 to p_2, v_1 must lie to the right of u_4 and v_2 must lie to the left of u_3. Starting at p_1, the path from p_1 to p_2 uses lh_2 or v_2, from there it must continue to rh_1 or v_1. This implies that H_1 or H_2 must be crossed again. So T' uses an edge of length $b + h$ in $p_1\bullet\!\!\sim\!\!\bullet p_2$, contradiction.

The only possibility left is that h_1 and h_2 are on the same horizontal carrier. Now only one of v_1 and v_2 can lie on u_4 c.q. u_3, otherwise T' would support the theorem.

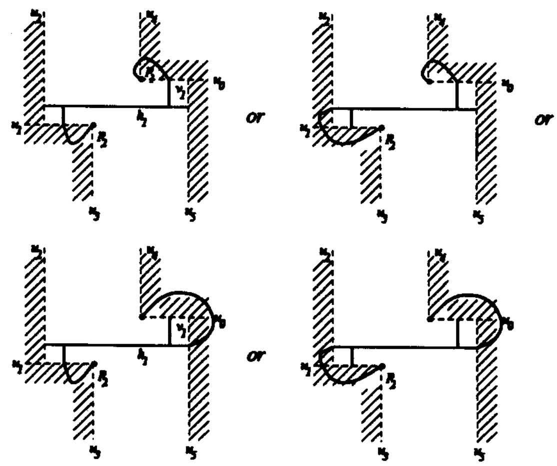


Figure 61: Slanted clustering, Z-shape: case 3, h_1 and h_2 between u_1 and u_0

All possible situation are depicted in figure 61. If v_1 (or analogously v_2) is used in $p_1\bullet\!\!\sim\!\!\bullet p_2$, situations a, b and c, then move v_1 to u_4. In situation a this creates a tree supporting the theorem. In situations b, c and d, rh_1 (or analogously lh_2) is used in $p_1\bullet\!\!\sim\!\!\bullet p_2$. The following transformations create, for these situations, a tree supporting the theorem. Move the part of h_1 which lies between u_4 and v_1 up to u_0, thus creating $p_1\bullet\!\!-\!\!\bullet v_{1_t}$. Then move v_1 to u_4. Furthermore rh_1 is now obsolete so T' is not a RMST. Contradiction.
This completes the proof of case 3 and of the theorem.
□

The following lemma is an instance of theorem 13.

Lemma 15 *If p_1 and p_2 have a carrier in common and $(H_1 \cup H_2) \cap V = \emptyset$, then a RMST T for V exists, which contains $p_i\bullet\!\!-\!\!\bullet p_{i+1}$.*

In this section several theorems have been proved with which line segments can be added before an approximation algorithm is applied. Note that the placement of one such a line segment can prohibit the placement of an other line segment. For example consider the four corner points of a square, see figure 62. Three sides of this square can be added using some of the previous theorems. Adding the fourth side would imply that one of the other sides would be removed.

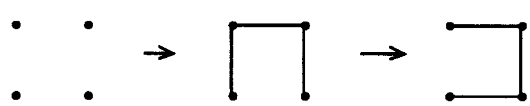


Figure 62: Clustering used repeatedly.

8 Optimizations

The proposed algorithm produces trees A where $\frac{|A|}{|T|} < \frac{3}{2}$. In most cases, there are some optimizations possible that will reduce this bound further. One of the possibilities is the application of slanted clustering. This can easily be done by creating the path proposed by the slanted clustering and checking if the resultant cycle contains a part of length greater than the inserted path. Trivially, the complexity of checking all possible slanted clusterings is $O(n^2)$.

The optimized tree sometimes has a very easily detectable form of suboptimality. This form of suboptimality occurs when it is possible to reverse one or more L-shapes and delete a resulting overlap. This can then be solved by replacing one or more corners by Steiner points and deleting a line segment.

Definition 1 *A RST T for a set V is corner-1-optimal if there is no RST T' for V with $|T'| < |T|$ that can be constructed from T by reversing a L-shape.*

Lemma 1 *Any RST T for a set V can be made corner-1-optimal in $O(n)$ time.*

Proof. A reversal of any L-shape can be accomplished in $O(1)$ and T contains at most $n-1$ corners.

Definition 2 *A RST T for a set V is corner-2-optimal if there is no RST T' for V with $|T'| < |T|$ that can be constructed from T by reversing two L-shapes.*

Lemma 2 *Any RST T for a set V can be made corner-2-optimal in $O(n)$ time.*

Proof. A reversal of any L-shape can be accomplished in $O(1)$ and T contains at most $n-1$ corners.

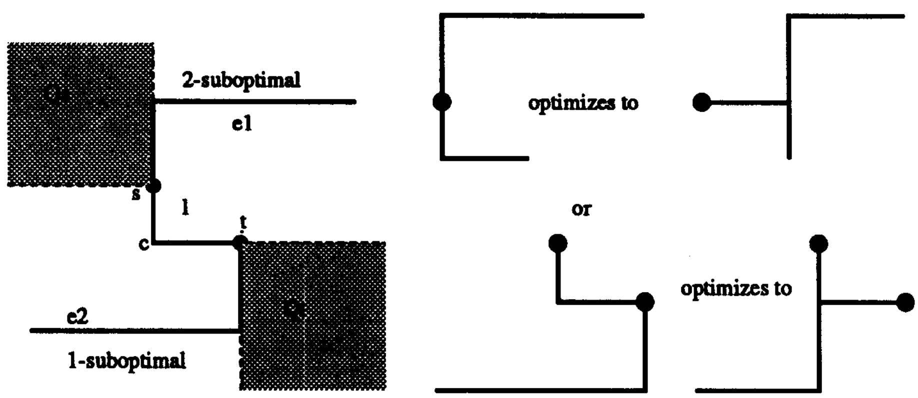


Figure 1: Corner suboptimality

9 The approximation algorithm

algorithm	Main	
input	A set of points V	
output	A RST A with $\frac{\|A\|}{\|T\|} < \frac{3}{2}$	
begin		
	sort(V)	
	build structure to contain RST	$O(n)$
	while contraction in some direction legal	$R\times$
	do contract in some direction	$O(n)$

```
        L:=contracted line segments                                 O(1)
        V:=contracted set                                           O(1)
        build RVD(V)                                                O(nlogn)
        for all pairs of neighbouring points in RVD(V)              O(n×)
            do  for all possible clusterings                            O(1×)
                    do  if corresponding rectangles empty                   O(n)
                            then add line segment to A                      O(1)
        for all pairs of neighbouring points in RVD(V)              O(n×)
            do  for all possible slanted clusterings                    O(1×)
                    do  if corresponding rectangles empty                   O(n)
                            then add pair to C                              O(1)
        X := A, a set of open line segments and points              O(1)
        construct RICH(X)                                           O(nlogn)
        construct RDAGVD(X)                                         O(n^2)
        while the forest in X still unconnected                     O(n×)
            do  add the shortest edge within RICH(X) to A               O(n)
                update RDAGVD(X)                                        O(n)
        for every corner in A                                       O(n×)
            do  if corner is 1- or 2-suboptimal                         O(1)
                    then reverse corner(s)                          O(1)
        for every pair in C                                         O(n×)
            do  if path between pair shortens A
                    then replace by path between pair               O(n)
        A := A ∪ L                                                  O(n)
end
```

R denotes the number of contractions performed.

```
algorithm  Split
input      A set of points V
output     A RST A with |A|/|T| < 3/2
begin
           compute RICH(V)                                          O(nlogn)
           split V on its articulation points into V_1, ..., V_a    O(n)
           for each of the sets V_i, i ∈ {1, ..., a}                O(n^2)
               do  A_i:=MAIN(V_i, m)
           A:=∪_{i∈{1,...a}} A_i                                    O(1)
end
```

Theorem 14 (MAIN) *The construction of an approximate RST A for a set V containing n points in the plane with RMST T, where A is a corner-2-optimal tree and $\frac{|A|}{|T|} < \frac{3}{2}$ can be performed in $O(n^2)$*

10 An example

In this section, the resulting approximate RST for a set of 37 points is presented. With the exception of the split on articulation points, every construction tool described in this article is applicable at least once.

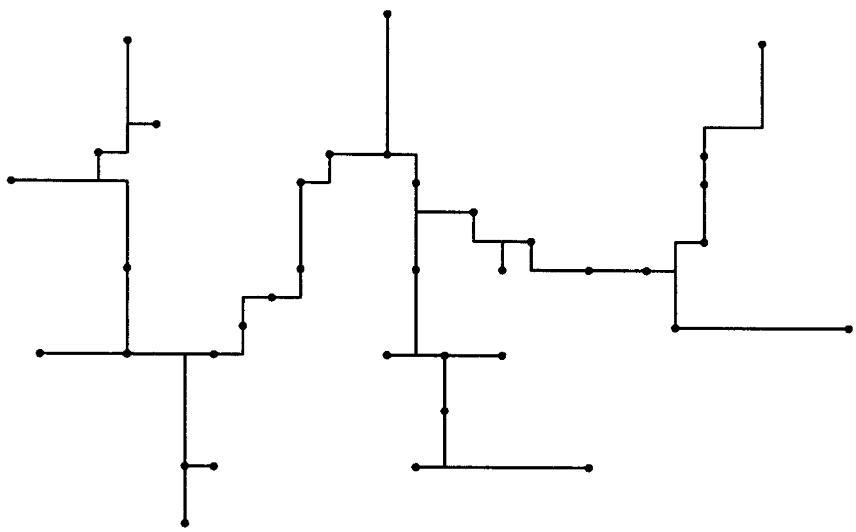


Figure 64: After application of 2-corner-optimality

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